PHY354 | Lagrangian $L = T - V$ | Energy $E = T + V$ | Euler Lagrange $\boxed{\frac{d}{dt}\left(\frac{dL}{d\dot{q}_i}\right) = \frac{dL}{dq_i}}$ → minimizes/extremizes Action

King Fung | Action $S = \int_{t_1}^{t_2} L[q(t), \dot{q}(t), t]\, dt$. NB: αL has same sol^n as L

Constrained Systems: ① if a coordinate can be expressed as $f(\text{coordinate}_2)$, or as a constant, substitute it in to $\mathcal{E}$-L eq^{n}s.

② if a coordinate isn't conserved, you can only substitute it into E or sol^{n}s of $\mathcal{E}$-L, **NOT $\mathcal{E}$-L itself.**

Cyclic Coordinates: (conserved) if $\frac{dL}{dq} = 0$, $\frac{dL}{d\dot{q}} = C$ w.r.t. Ex. $L = \frac{1}{2}m\dot{x}^2$ → $m\dot{x}$ is constant in time

Noether's Theorem: if a transformation $Q_i(t,\lambda)$, $(Q_i(t,0) = q_i)$, & $\frac{dL}{d\lambda}(Q_i, \dot{Q}_i, t) = \frac{dF}{dt}$, or more commonly ↳ $F(Q, \dot{Q}, t)$

$$\frac{d}{dt}\underbrace{\sum \frac{dL}{d\dot{q}_i}\frac{dQ_i}{d\lambda} - F}_{\text{conserved quantity}} = 0$$

→ usually $\sum \frac{dL}{d\dot{q}_i}\frac{dQ_i}{d\lambda} = C \in \mathbb{R}$

① define $Q_1(t, \lambda_1)$ ② define all other Q_i w/ transformation, proportional to λ_1 by $\alpha, \beta, \ldots$

③ plug into derivative of L → $\frac{dL}{dt}$ & equate to $\frac{dF}{dt}$ or 0

④ find α, b, etc. ⑤ compute $\sum \frac{dL}{d\dot{q}_i}\frac{dQ_i}{d\lambda}$, which is conserved. * External forces may affect this

Conserved Quantities: try a translation of $q_i + \varepsilon$ if $U(q)$ doesn't change, a quantity

$x, y, z \Rightarrow P_{x,y,z}$ $\phi \Rightarrow M_z$ → integrals of motion (the cons'd quantity)

Max # of Integrals of Motion $= 2d - 1$, d = degrees of freedom.

Central Potential $U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$ $\vec{r} = \vec{r}_1 - \vec{r}_2$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ $\vec{R}_{COM} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$ $r_2 = \frac{-m_1}{m_1+m_2}\vec{r}$, $r_1 = \frac{m_2}{m_1+m_2}\vec{r}$ } in COM frame.

↳ conserved $\vec{H}$ & $\vec{M}$ $L = \frac{1}{2}\dot{\vec{r}}^2(\mu) - U(|\vec{r}|)$

$V_{eff} = \frac{M_z^2}{2mr^2} + U(r)$ ← centripetal energy

M const, $\vec{M}_z = mr^2\dot{\phi}^2 = \text{const}$; $M = \vec{r} \times \vec{p}$

↳ for $U(r) = \frac{-k}{r}$, $\vec{A} = \vec{p} \times \vec{M} - mk\hat{r}$ is conserved

Vector Identities

$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$

$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$

$e = \frac{A}{mk}$ → $e = 0$ is circular, $e > 1$ hyperbola, $0 < e < 1$ elliptical

Reducing to Quadratures: $q_2 = \int_{q_{1_0}}^{q_{1_f}} ? \, dq_1$

↳ solve for $\frac{dQ_1}{dq_2}$, integrate $\frac{1}{\frac{dq}{dq_2}}$ over known bounds

Mechanical Similarity transforms given α, β, … if α & β are equal such that $L' = C \times L$ (α for r, β for t.)

for $q_1 = \alpha q_1$, $q_2 = \beta q_2$ … ↳ $L = T - V$ → given V st $V(\alpha r_1, \ldots \alpha r_n) = \alpha^k V(r_1, \ldots r_n)$

$= \frac{\alpha^2}{\beta^2}T - \alpha^k V$ $\frac{\alpha^2}{\beta^2} = \alpha^k$ $\beta = \alpha^{1 - k/2}$

Good to know

$F_{grav} = -\frac{G M_1 M_2}{r^2}\hat{r}_{12}$ or $\hat{r}_{21}$

$U_{grav} = mgh$

$U_{spring} = \frac{1}{2}kx^2$

Kepler: $\frac{T^2}{r^3} = \frac{4\pi^2}{GM}$ (T: orbit period; r: semi major axis of ellipse, or radius of circle) $\propto \frac{1}{M_{star}}$

$ds = \sqrt{dx^2 + dy^2 + dz^2}$

$\sum_{a\in ?} m_a \vec{r}_a \Rightarrow M R_{com}$, ↳ $= \sum_{a\in G} m_a$

$x = r\cos\phi$	$x = \rho\sin\theta\cos\phi$
$y = r\sin\phi$	$y = \rho\sin\theta\sin\phi$
$z = z$	$z = \rho\cos\theta$

$M_x = (z\dot{y} - y\dot{z})m$

$M_y = (z\dot{x} - x\dot{z})m$

$M_z = (x\dot{y} - y\dot{x})m$

Worked Examples

$$\frac{y'}{\sqrt{1+y'^2}} = c \qquad y' = \sqrt{\frac{c^2}{c^2+1}}$$

1. Gallilean Invariance:

$$\vec{r}_k = \vec{r}_k + \lambda \vec{u} t \qquad , \vec{u} = \text{const}$$

$$L(\vec{r}, \dot{\vec{r}}) = \sum \tfrac{1}{2} m_k (\dot{\vec{r}}_k + \lambda \vec{u}) - V$$

$$= L + \left(\sum m_k \dot{\vec{r}}_k\right)(\vec{u}\lambda) + O(\lambda^2)$$

$$= L - \lambda \frac{d}{dt}\left(-\vec{u} \sum_k m_k \vec{r}_k\right) + O(\lambda^2)$$

QUALITATIVE ORBITS
↳ Find $V_{eff}(r)$, define motion from there

$$\text{so } \frac{\delta L(Q)}{\delta \lambda} = \frac{d}{dt}\left(-\vec{u} \sum m_k \vec{r}_k\right) + \sim$$

$$\therefore \sum \frac{dL}{dq} \times \frac{dQ}{d\lambda} F = \sum m_k \dot{\vec{r}}_k \cdot \vec{u} t) - \vec{u}; M\vec{R}_{com}$$

$$= \vec{P}_{sys} \vec{u} t - \vec{u} M \vec{R}_{com} \quad \to u \text{ is arbitrary}$$

$$= \vec{P}_{sys} t - M \vec{R}_{com}$$

Quantitative → solve $\frac{dr}{dt} \to t = \int_{r_{min}} \frac{dt}{dr} dr$

$$\frac{d\phi}{dt} = \frac{d\phi}{dr}\frac{dr}{dt} \Rightarrow \phi = \int \frac{dt}{dr} dr^2$$

$$= \frac{M}{mr^2}$$

2. Scaling

$eqn_1 = const_1 \qquad eqn = const_2 \quad \to$ divide one by the other.

3. Atwood: $L = \frac{1}{2}\left(m_1 \dot{x}_1^2 + m_2(\dot{x}_1^2 + \dot{x}_2^2 - 2\dot{x}_1\dot{x}_2) + m_3(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\dot{x}_2)\right) + m_1 g x_1 + m_2 g(x_2 - x_1) - m_3(x_1 + x_2)$

$$V \Rightarrow V + m_1 g \varepsilon_1 + m_2 g(\varepsilon_2 - \varepsilon_1) - m_3 g(\varepsilon_1 + \varepsilon_2)$$

$$x_1 = x_1 + \varepsilon_1 \qquad x_2 = x_2 + \varepsilon_2$$

require $V \to V$, solution $\frac{\varepsilon_2}{\varepsilon_1} = \frac{m_3 + m_2 - m_1}{m_2 - m_3}$

Noethers: $\left(\frac{dL}{d\dot{x}_1} + \frac{\varepsilon_2}{\varepsilon_1}\frac{\partial L}{\partial \dot{x}_2}\right)$ is conserved.

4 Perturbation Arguments → $U(r_0 + \varepsilon)$ → taylor expand: $O(\varepsilon^2)$ terms disappear

$$U'(r_0) = 0$$

5. Reducing to quadratures $L = \frac{1}{2} m \dot{x}^2 - U(x) \qquad E = \frac{1}{2} m \dot{x}^2 + U(x)$

$$\frac{dx}{dt} \dot{x} = \sqrt{\frac{2(E - U(x))}{m}} \Rightarrow dt = \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{2(E - U(x))}}$$

$$T_{transit} = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - U(x))}}$$

$$\phi = \int \frac{M \, dr/r^2}{\sqrt{2m[E - U(r)] - M^2/r^2}}$$