PHY 485 KCBF

Essentials (Math)

$f(x)_{@a} = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$

ex: $e^x_a = e^a\left(1 + (x-a) + \frac{(x-a)^2}{2} \dots\right)$ / $\sqrt{1+x} = 1 + \frac{1}{2}x + \dots$

$\sin x_0 = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$ / $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

$\sum_{i=0}^{\infty} x^n = \frac{1}{1-x}$, $\sum_{i=0}^{N-1} x^n = \frac{1-x^N}{1-x}$, $\int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3}$

$\int e^{-\alpha x^2 + bx + c} dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a} + c}$

ODEs $\frac{dy}{dx} = ay \Rightarrow \ln y = ax + C$

$\frac{dy}{dx} = \frac{g(x)}{h(y)} \Rightarrow \int h(y)dy = \int g(x)dx + C$

Techniques

$\frac{dy}{dx} + h(x)y = g(x) \Rightarrow \mu(x)y = \int \mu(x)g(x)dx + C$

where $\mu(x) = e^{\int h(x)dx}$

HH Eqn $(\nabla^2 + k_0^2)\, g(\vec r - \vec r') = \delta^{(3)}(r - r')$

$\hookrightarrow g(\vec r - \vec r') = -\frac{1}{4\pi} e^{ik_0 R}$

$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(r, r', t, t') = \delta^3(\vec r - \vec r')\,\delta(t - t')$

$\hookrightarrow G(\vec r - \vec r', t, t') = -\frac{1}{4\pi}\frac{\delta(t - t' - R/c)}{R}$

Essentials (Physics)

Maxwell's EQN $\nabla\cdot\vec D = \rho_f$, $\nabla\times\vec H - \frac{d\vec D}{dt} = \vec J_f$

$\nabla\cdot\vec B = 0$, $\nabla\times\vec E + \frac{\partial\vec B}{\partial t} = 0$

where $\vec D = \varepsilon_0 \vec E + \vec P$, $\vec B = \mu_0 \vec H$

Also: $\vec B = \nabla\times\vec A = \frac{1}{c}(\vec A\times\hat n)$, $\vec E = ((\vec A\times\hat n)\times\hat n)$

$A \approx \frac{\mu_0}{4\pi r}\dot{\vec d}$

$S = \frac{1}{\mu_0}(\vec E\times\vec B)$

CoM $\vec R = \frac{m_e \vec r_e + m_n \vec r_n}{\sum m}$, $\vec x = \vec r_e - \vec r_n$

$M = \sum m$

$E_\gamma = h\nu$, $k = \frac{2\pi}{\lambda}$, $\omega = \frac{2\pi}{T} = 2\pi\nu$

$p = \frac{h}{\lambda} = \frac{h\nu}{c}$, $\nu\lambda = \lambda f = V$

$\gamma \approx 10^{10}\,s^{-1}$, $\omega \approx 10^{13}\,s^{-1}$

$\boxed{\frac{N_2}{N_1} = e^{-h\nu/k_B T}}$

$\tilde H(\omega) = \frac{1}{2}\delta(\omega) - \frac{1}{2\pi i\omega}$

$I = \frac{1}{2}c\,\varepsilon_0 E_0^2$

$\int dx\,dy = \int r\,dr\,d\theta$

$v_\phi = \frac{c}{n_r(\omega_0)}$

① Absorption, Scattering of Light

Force on a dipole (Ext Field) $M\frac{d^2\vec R}{dt^2} = e(\vec x\cdot\nabla_R)\vec E(\vec R, t)$

Spacing due to Ext field $\frac{d^2 x}{dt^2} = \frac{e}{m}\vec E(\vec r_e, t) + \frac{1}{m}F_{en}$

Static $m\ddot x = F_{en}$ Com

Damped Osc $\ddot x(t) + \gamma\dot x(t) + \omega^2(t) = \frac{e}{m}E(q, t)$ EOM

Start $\gamma\dot x \omega x \in$ $\hookrightarrow$ $x(t) = A\, e^{-\gamma t/2}\cos(\omega_1 t + \phi)$, $\hookrightarrow \omega_1^2 = \omega_0^2 - \left(\frac{\gamma}{2}\right)^2$

KGF: $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E = \alpha\gamma\dot{\vec x}$ energy $\mathcal{E} = \frac{mA^2}{2}e^{-\gamma t} = \frac{m}{2}(\dot x^2 + \omega_0^2 x^2)$

Power out of dipole: Start: $\vec S = \frac{1}{\mu_0}\vec E\times\vec B$, $\int S\, d\Omega$

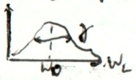
$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c^3}(\ddot x)^2$

$= \frac{1}{4\pi\varepsilon_0}\left(\frac{2}{3}\right)\left(\frac{e^2}{c^3}\right)\left(\frac{\omega_0^2}{m}\right)\mathcal{E}$

Spacing response due to light

Start: EoM, $X(t) = \sum \frac{e}{m} E_0 \mathrm{Re}\left\{\frac{e^{i(\omega_L t + \phi)}}{-\omega_L^2 - i\omega_L\gamma + \omega_0^2}\right\}$

$E = \frac{e}{m}\cos(\phi + \omega_L t)$, Fourier transform



Planck: $\rho(\nu) = \frac{8\pi h\nu^3/c^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$ $\rightarrow (\beta_K : \dots)$

$\frac{W}{m^2\,Hz}$

Energy Absorbed from light

Start: $\frac{d\mathcal{E}}{dt} = \vec F\cdot\vec v$

$= \vec\varepsilon\, e E_0\cos(\omega_L t + \phi)\frac{d\vec x}{dt}$

$\frac{d\mathcal{E}}{dt} = \frac{(eE_0)^2}{m}\frac{\omega L^2\gamma/2}{(\omega_0^2 - \omega_L^2)^2 + (\gamma\omega_L)^2}$ and then math

Planck RATE EQN

$\frac{dN_1}{dt} = -\frac{dN_2}{dt} = A_{21}N_2 - \frac{A_{21}c^3}{8\pi h\nu_0^3}N_1\rho(\nu_0) + \frac{A_{21}c^3}{8\pi h\nu^3}N_2\rho(\nu_0)$

$= \frac{\sigma(\nu)}{h\nu}\left(N_2 - \frac{g_2}{g_1}N_1\right)I + A_{21}N_2$, where

$\sigma(\nu) = \frac{\lambda^2}{8\pi}A_{21}\,S(\nu)$ ← normalized line absorption

$I = \rho(\nu_0)\cdot c$ [can add $\frac{g_1}{g_2}N_2$ for both]

BROADENING

↳ homogeneous: Collision, radiative

↳ inhomogeneous: Doppler, Stark / Zeeman

Dop: $S(\nu) = \frac{1}{\sqrt{\pi}\,\delta\nu_D}\, e^{-\frac{(\nu - \nu_0)^2}{2\delta\nu_D}}$

↳ $\delta\nu_D = \sqrt{\frac{k_B T}{M}}\frac{\nu_0}{c}$

Voight: $S_{eff}(\nu) = \frac{1}{\pi^{3/2}}\frac{b^2}{\delta\nu_D}\int_{-\infty}^{\infty}\frac{dy\, e^{-y^2}}{(y - x)^2 + b^2}$

$y = \dots$, $x = \frac{\nu_0 - \nu}{\delta\nu}$, $b = \dots$

Linear Response: $P(t) = \varepsilon\int_{-\infty}^{\infty}\chi(t - t')E(t')dt'$ → causal

$\nabla\times\nabla\times\vec E + \frac{1}{c^2}\frac{\partial^2\vec E}{\partial t^2} = 0$

$-\nabla^2 E + \nabla(\nabla\cdot E) + \mu_0\frac{d^2 P}{dt^2}$ $\boxed{n(\omega) = n_r(\omega) + i n_i(\omega)}$

FT ⇒ $\left(\nabla^2 + \frac{\omega^2}{c^2}(1 + \tilde\chi(\omega))\right) = 0$ $\boxed{= \sqrt{1 + \tilde\chi(\omega)}}$

Kramer's Kronig

$\tilde\chi_1(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tilde\chi_2(\omega')}{\omega' - \omega}d\omega'$

$\tilde\chi_2(\omega) = \frac{-1}{\pi}\int_{-\infty}^{\infty}\frac{\tilde\chi_1(\omega)}{\omega' - \omega}d\omega'$

$\tilde\chi(\omega) = \tilde\chi_1(\omega) + i\tilde\chi_2(\omega)$

② Laser Oscillation

Gain: $g = \sigma\left(N_2 - \frac{g_2}{g_1}N_1\right)$ ★

For a beam in z dir,

↳ $\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)I = gI$

SS: $I = e^{gz}$

For SS operation:

$r_1 r_2 e^{2g l} = 1 \Rightarrow \boxed{g_{th} \simeq \frac{1 - r_1 r_2}{2l}}$

2 level Rate Eqn $\Phi = \frac{I}{h\nu}$

$\dot N_1 = A_{21}N_2 + g\Phi$

$\dot N_2 = -A_{21}N_2 - g\Phi$

Given Φ const & $N_1 + N_2 = N$

$\dot N_2 = -A_{21}N_2 - \sigma(2N_2 - N)\Phi$

$= -(A_{21} + 2\sigma\Phi)N_2 + \sigma\Phi N$

$\dot N_2 + \alpha N_2 = \sigma\Phi N$

$\int dt \dots = e^{-\alpha t}N(0) + N_2(t) = \frac{\sigma\Phi N}{\alpha}(e^{\alpha t} - 1)$

$N_2(t) = \left[N_2(0) + \frac{\sigma\Phi N}{\alpha}\right]e^{-\alpha t} + \frac{N\sigma\Phi}{\alpha}$

$A_{21} \gg \sigma\Phi$ ($N_2 \ll N$) } ss sol'n of N_2

$A_{21} \ll \sigma\Phi$ ($N_2 \approx \frac{1}{2}N$) } sign regime

gain analysis (2 level)

$g = \sigma(N_2 - N_1)$

$= -\frac{N A_{21}\sigma}{A_{21} + 2\sigma\Phi} = \frac{-N\sigma}{1 + \frac{2\sigma}{A_{21}}\Phi}$

$g = \frac{g_0}{1 + \Phi/\Phi_{sat}} \Rightarrow g_0 = -N\sigma$, $\Phi_{sat} = \frac{A_{21}}{2\sigma}$

3 level System

$\dot N_1 = A_{21}N_2 + \sigma(N_2 - N_1)\Phi + P(N_3 - N_1)$

$\dot N_2 = -A_{21}N_2 - \sigma(N_2 - N_1)\Phi + A_{32}N_3$

$\dot N_3 = -P(N_3 - N_1) - A_{32}N_3$

Set all $= 0$ (S.S.) $N_3 \ll N_1$

$\dot N_1 = (A_{21} - P)N_2 + (\sigma\Phi + P)(N_1 \dots)$

↳ compare to above, we get

$g = \frac{(P - A_{21})N}{P + A_{21}}\sigma$, $\Phi_{sat} = \frac{(A_{21} + P)}{2\sigma}$

$\boxed{P = \frac{A_{32}}{A_{32} + P}P}$

Resonance Effects

$g_0(\nu) = \frac{g_0(\nu_0)}{1 + \left(\frac{\nu - \nu_0}{\delta\nu_0}\right)^2}$ $\Phi_{sat}(\nu) = \Phi(\nu_0)\{\dots\}$

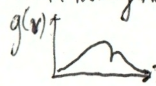
$g(\nu) = \frac{g(\nu_0)}{1 + \left(\frac{\nu - \nu_0}{\delta\nu_0}\right)^2 + \Phi/\Phi_{sat}}$

Spatial Hole Burning

$g = \frac{g_0(\nu)}{1 + \frac{2\Phi}{\Phi_{sat}}\sin^2(kz)}$, $I = 2I_0\sin^2(kz)$

Spectral Hole Burning

↳ in homogeneous broadening (saturates more evenly)



Gain Clamping

$\frac{d\Phi_{cav}}{dt} = \frac{cl}{L}g(\nu)\Phi - \frac{c}{2L}(1 - r_1 r_2)\Phi$

SS: $g_{Th} = \frac{(1 - r_1 r_2)}{2l}$

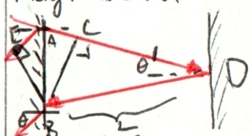
In the cavity → $\Phi = \Phi_{sat}\left(\frac{g_0}{g_{th}} - 1\right)$

$I_{out} = t\,\Phi_- + t\,\Phi_+$

$= \frac{t}{2}\Phi = l\,\Phi_{sat}(g_0 - g_{th})$

$= \frac{lN(P - A_{21})}{2}$

Fabry Perot Etalon ③



Obtain admitted rays:

$\delta = 2\pi\frac{\nu}{\Delta\nu}$ (FSR)

where $\Delta\nu = \frac{c}{2L n\cos(\theta')}$

Stokes reciprocity Relations

$r^2 + tt' = 1$

$t(r + r') = 0$

Start: reverse, compare to original ray

r_{eff} Stokes: $-r_1 + \frac{t_1 e^{i\delta}(r_1')^2 t_2' + r_2 t_1 t_1'}{1 - r_1' r_2' e^{i\delta}}$

$|r_{el}|^2 = \frac{4R\sin^2\frac{\delta}{2} + (r_1' - r_2')^2}{(1 - R)^2 + 4R\sin^2\left(\frac{\delta}{2}\right)}$

$R = r_1' r_2'$

w/ Start: $\sum$ reflected rays

Etalon Parameters

$T_e = 1 - |r_e|^2 = 1 - R_e$

$= \frac{T_{max}}{1 + F\sin^2\left(\frac{\pi\nu}{\Delta\nu}\right)}$

via $F = \frac{4R}{(1 - R)^2}$, $T_{max} = 1 - \left(\frac{r_1' - r_2'}{1 - r_1 r_2}\right)^2$

HWHM $\delta\nu = \frac{\Delta\nu}{\pi\sqrt F}$

$\mathcal{F} = \frac{\pi\sqrt F}{2} = \frac{\pi\sqrt R}{1 - R}$

EF: E(t) and R(t) from Stokes...

Intra cavity

$I(z) = I_0\left\{1 - \frac{\frac{4r_2'}{(1 + r_2')^2}\sin^2\left(\frac{\pi\nu}{\Delta\nu}z\right)}{1 + F\sin\left(\frac{\pi\nu}{\Delta\nu}\right)}\right\}$

$r_1' = 1 - \xi\ \ \xi'$

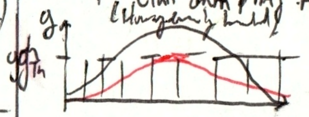
Allow longitudinal modes $\nu = m\Delta\nu$ FP:

Single mode Operation

↳ Op 1: limit the bandwidth

$\frac{c}{2L} \gg \Delta\nu_g$: $L \ll \frac{c}{2\Delta\nu_g}$

↳ Op 2: Gain damping / resonance (changing the bandwidth)



→ More robust: Add tuned output cavity

Band width: dominated by Spont Emission

$\Delta\nu \geq \frac{N_2}{\Delta N_t}\frac{h\nu_0(4\pi\Delta\nu_c)^2}{P_{out}}$

$(N_2 - N_1)$

$\Delta\nu_{cavity} = \frac{1}{2\pi}\frac{c\, g_{th}}{2L}$

$E(r, t) = \varepsilon(t)$

$\mathcal{F}(\varepsilon)$ 1) $N\varepsilon$: $R(0) \geq 0$, coh

so not unique in phase

but if $S(\omega)\ \&\ (\omega_i - \omega_2)$

$= \langle\varepsilon\,\varepsilon^*\rangle$

$\mathcal{F}(R(\tau)) = S(\omega)$

④ Diffraction Theory

Scalar ⇒ $E(r) = U(\vec r)e^{-i\omega t}$

Gauss: $\nabla\cdot E = 0$ $(\nabla^2 + k_0^2)U = 0$

The propagation Prob.: $u(x,y,0) \to u(x,y,z)$

$$\boxed{(\nabla^2 + k_0^2)\, g(\vec r' - \vec r) = \delta^3(r - r')} \quad (4a)$$

↳ $g = -\frac{1}{4\pi}\frac{e^{ik_0 R}}{R}$ (4b)

Then, using Divergence Thm:

$\int\!\!\int_V (\nabla\cdot\vec F)\, d^3r = \oint_S \vec F\cdot d\vec S$

w/ $F = u\nabla g - g\nabla u$

$\nabla F = u\nabla^2 g - g\nabla^2 u$

Thus:

$$\oint_S (u\nabla g - g\nabla u)\cdot d\vec S = \begin{cases} u(r') & (r\in V) \\ 0 & o/w \end{cases} \quad (4c)$$

HUYGENS!

Solved for the hemispherical case. $R = |\vec r_0 - \vec r|$

↳ No contribution from shell.

From (4c) & 4b

$U(\vec r_0) = -\int dx\,dy\,\left(\frac{\partial U}{\partial z} g - g\frac{\partial U}{\partial z}\right)$

$U(\vec r') = -\int dx\,dy\,\left(u\frac{\partial g}{\partial z} - g\frac{\partial u}{\partial z}\right)$

$\frac{\partial R}{\partial z} = -\frac{z}{R}$ yields,

$$U(\vec r_0) = \frac{-1}{2\pi}\int\!\!\int dx\,dy\, \frac{z_0}{R}\left(ik - \frac{1}{R}\right)\frac{e^{ikR}}{R}\, u(x,y,0) \quad (4D)$$

RAYLEIGH SOMMERFIELD FORMULA

For a disk: ($E \propto$ step(a)) polar coordinates, & $E = \mathcal{E}_0$

CoV from $\rho \Rightarrow R = \sqrt{\rho^2 + z_0^2}$

leads to: $u(0,0,z_0) = -\frac{\mathcal{E}_0 z_0}{\sqrt{a^2+z_0^2}}\, e^{ik\sqrt{a^2+z_0^2}}$

Fraunhofer Far Field: $R = r_0\hat n$

→ From (4D) use:

① $\frac{z}{R} \approx n_z$ ($r_0 \gg R$)

② $r_0 \gg \lambda$, $ik \gg \frac{1}{R}$

to get $R = r_0 - n\cdot r_{x,y\text{ on plane}}$

& $U(r_0\vec n) = \frac{1}{2\pi r_0} k e^{-ik_0 r}\, n_z (2\pi)^2\, \tilde U_0(-k_0\vec n)$

$\tilde U_0(\vec k) = \frac{1}{(2\pi)^2}\int\!\!\int dx\,dy\, U(x,y,0)\, e^{-ik\cdot r}$

$|\vec r_0 - \vec r| = \sqrt{(\vec r_0 - \vec r)^2}$

Fresnel Diffraction (Parax)

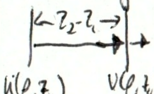
Using (4D) & $z_0 \gg r$ } assumpt

$k_0 \gg \frac{1}{R}$, $\frac{z_0}{R} \approx 1$

$R \approx z_0 + \frac{1}{2z_0}\{(x_0-x)^2 + (y-y_0)^2\}$ & get

& $U(\rho_2, \rho_1) = \frac{ik}{2\pi z_0}\, e^{-ikz_0}\int d\rho_2\, U(\rho_1)\, e^{\frac{ik}{2z_0}(\rho_2' - \rho_1)^2}$

Kernel Analysis



$U(\vec\rho, \vec z_2) = \int_{-\infty}^{\infty} d^2\rho\, U(\vec\rho_1, z)\, K(\rho_2, \rho_1, \Delta z)$

$U(\rho, z_1)$ $\quad U(\rho, z)$

$K_{\text{parax}}(\rho, z) = \frac{ik}{2\pi z}\, e^{ikz}\exp\left\{\frac{ik_0}{z}\rho^2\right\}$

Lens analysis:

$T(\rho) \approx z_2 - z_1 + R_2 + R_1 - \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\rho^2$

$OPL(\rho) \approx n_{\text{air}}[T(0) - T(\rho)] + T(\rho)\, n_{\text{glass}}$

$= \left(nT(0) - \frac{n-1}{2}\left(\frac{1}{R} - \frac{1}{R}\right)\rho^2\right)$

$K_{ls}(\rho, z) = -e^{iknT(0)}\, e^{\frac{-ik}{2f}\rho^2}\, \delta(\vec\rho)$

ABCD Matrices (paraxial)

$M\begin{pmatrix} A & B \\ C & D\end{pmatrix}$: $\det M = 1$, $M_{sys} = M_3 M_2 M_1$

Kernel: $B \neq 0$: $K(\rho_1, \rho_2) = \frac{-ik}{2\pi B}\cdot\exp[ikL]$

$\times \exp\left[\frac{ik}{2B}(A\rho_1^2 - 2\rho_1\rho_2 + D\rho_2^2)\right]$

if $B = 0$ use $AD - BC = 1$ to get rid of $0/0$s in the expont then treat as δ (argument)

Ex, Prop: straight forward $\begin{bmatrix} 1 & d \\ 0 & 1\end{bmatrix}$

Ex₂ lens: chg. vary ⇒ $\frac{A-1}{2}(\rho_1 + \rho_2)^2 + \left(\frac{A+1}{2}\right)(\rho_1 - \rho_2)^2$

$\frac{A-1}{2} \Rightarrow \frac{C}{4(A+1)} \Rightarrow \delta$ thereof $\begin{bmatrix} 1 & 0 \\ -1/f & 1\end{bmatrix}$

Ex₃ Imaging: $\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1\end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix} = \begin{bmatrix} -M & 0 \\ -1/f & \frac{1}{M}\end{bmatrix}$ when $\frac{1}{a} + \frac{1}{b} = \frac{1}{f}$

↳ $A = \frac{1}{D}$, so factor & get $(A\rho_1 + \rho_2)^2$ then δ along

Ex₄ Imaging, but $f = a = b$ $\begin{pmatrix} 0 & f \\ -1/f & 0\end{pmatrix}$

↳ $K = \frac{-ik}{2\pi f}\, e^{ikL}\exp[i f\cdot\rho_1]$, $f = \frac{k_0}{f}\rho_2$

Gaussian Beams!

$(\nabla^2 + k^2)E(\vec r) = 0$

$E = \mathcal{E}(r)e^{ikz}$

w/ $\left|\frac{\partial\mathcal{E}}{\partial z}\right| \ll k\mathcal{E}_0$

sch $\nabla_\perp^2\mathcal{E}_0 = 2ik\frac{\partial\mathcal{E}}{\partial z}$

Prototype sol: $\mathcal{E} = A\exp\left(\frac{ik\rho^2}{q}\right)\exp[ip(z)]$

Sol^n: $\frac{dq}{dz} - 1 = 0$ $\quad \frac{dP}{dz} = 0$

$q(z) = z + q_0$, if $p_0 = 0$, $p = i\ln\left(\frac{q_0 + z}{q}\right)$

full sol^n

$$\mathcal{E}_0(x,y,z) = \frac{Ae^{-i\phi(z)}}{\sqrt{1 + z^2/z_0^2}}\exp\left[\frac{ik(x^2+y^2)}{2R(z)}\right]\exp\left[\frac{-(x^2+y^2)}{W(z)^2}\right]$$

where: $\frac{1}{q} = \frac{1}{R} + \frac{i\lambda}{\pi W^2}$

$\boxed{q_0 = -iz_0}$ (IV)

$R(z) = z + \frac{z_0^2}{z}$ $\quad W(z) = W_0\sqrt{1 + (z/z_0)^2}$

$\tan\phi_{\text{Gouy}} = z/z_0$ $\quad W_0 = \sqrt{\frac{\lambda z_0}{\pi}}$

$|A| = \sqrt{\frac{2P}{c\,\mathcal{E}_0\left(\frac{\pi w_0}{2}\right)}}$

z_0: rayleigh range. $\quad W_0$: beam waist

How to find: R, W from q: $\frac{1}{q}$, rationalize, split, invert

ϕ from p: ⇒ $e^{ip} = \frac{1}{1 + i z/z_0}$, split into mag & phase.

$\phi_{\lim z\to\infty} = \frac{\pi}{2}$

A: $\int\!\!\int_{xy} \frac{1}{2}\mathrm{Re}\{\vec E \times \vec H^*\}$

↳ $E, H = \mathcal{E}_0(x,y,z)e^{ikz}$

$H = c\epsilon_0\,\mathcal{E}_0 e^{ikz}$

$\star\, Q\int_{xy}\exp\left[-\frac{2x^2 + 2y^2}{w(z)^2}\right] = \frac{\pi}{2}w^2 Q$ $\quad z = w_0\sqrt{1 + ...}$

$= P$

$Q = \frac{c\epsilon_0}{2}\,\frac{|A|}{1 + z^2/z_0^2}$

$\star$ @ waist, $q_i = -iz_0$

$R = \infty$

Mode Frequencies: $\nu_{q,n,m} = \frac{c}{2L}\left(q + \frac{1}{\pi}(n+m+1)\cos^{-1}(g_1 g_2)\right)$

ABCD Law for Gauss Beams

$$q_o = \frac{Aq_i + B}{Cq_i + D}$$

(proof requires $\det M = 1$)

Findings: focus @ a lens

$f_{\text{new}} \neq \frac{f}{\left(\frac{f}{z_0}\right)^2 + 1}$

$W_{0,\text{new}} = W_0\frac{f/z_0}{\sqrt{1 + (f/z_0)^2}}$

[fix eq, w/ rep. math]

$R \to \infty$ @ waist

Transverse Modes

Solves: $\frac{\partial^2 f}{\partial u^2} - 2u\frac{\partial f}{\partial u} + 2nf = 0$

Ex: $A\frac{W_0}{W(z)}H_n\left(\frac{\sqrt2 x}{W(z)}\right)H_m\left(\frac{\sqrt2 y}{W(z)}\right)$

$\exp\left[i(kz - (n+m)\arctan(z/z_0))\right]$ etc.

Stability

$g_1 = 1 - \frac{L}{R_1}$, $g_2 = 1 - \frac{L}{R_2}$

$0 < g_1 g_2 < 1$ (condition)

$g_1 g_2 = 1$ → quasi-stable

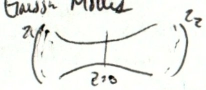
Deriv: $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ of RT chg. of P, R, P, R

$M^N = \frac{\sin N\theta}{\sin\theta}M - \frac{\sin(N-1)\theta}{\sin\theta}I$

$\cos\theta = \frac{A+D}{2}$

$A + D < 2$

Gauss Modes



$z_1 + \frac{z_0^2}{z_1} = -R_1$ $\quad z_2 + \frac{z_0^2}{z_2} = R_2$

$z_2 - z_1$:

$z_1 = \frac{-Lg_2(1 - g_1)}{g_1 + g_2 - 2g_1g_2}$ $\quad z_2 = L + z_1$

$z_0^2 = \frac{L^2 g_1 g_2(1 - g_1 g_2)}{...}$

$W_{1,2}^2 = \frac{\lambda L}{\pi}\sqrt{\frac{g_{1}}{g_2(1 - g_1 g_2)}}$

⑤ Random Processes & Coherence

$p(x_1, t)\,dx_1$ = prob $x \in [x_1, x_1 + dx_1]$ @ t

→ stationary process: t_0 is irrelevant, Δt matters

↳ reduce 1 in dim: $p(x_1, x_2, t_1, t_2) = p(x_1, x_2, t_2 - t_1)$

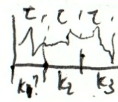
Ensemble avgs:

$\langle F(X_1(t_1), \ldots)\rangle = \int\cdots\int_{-\infty}^{\infty} F(x_1, \ldots)\, p(x, \ldots, t, \ldots)\,dx$

$\langle F(x(t))\rangle = \int_{-\infty}^{\infty} F(x)\,p(x,t)\,dx$

Time averages

$\overline{F[{}^k x(t)]} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} F({}^k x(t))\,dt$



→ EA & TA can be equal as $T \to \infty$

→ for STATIONARY & ERGODIC process

Analytic:

$\mathcal{Z}(t) = 2\int_0^\infty \hat X(\nu)\, e^{i\omega t}$

$X(t) = \mathrm{Re}(\mathcal{Z}(t))$, $\quad y(t) = \int_{-\infty}^{\infty}\frac{x(t)}{t - t'}$;

Correlation Fn.

$\Gamma(\vec r_1, \vec r_2, \tau) = \langle \mathcal{E}^*(\vec r_2, t - \tau)\mathcal{E}(\vec r_1, t)\rangle$

if Erg $= \overline{\mathcal{E}(\vec r_1, t+\tau)\mathcal{E}(\vec r_2, t)}$

Prop: $\Gamma^*(2, 1, -\tau) = \Gamma(1, 2, \tau)$ Hermitian

$\Gamma(r_1, r_1, 0) \geq 0$, $|\Gamma(r_1, r_2, 0)| < \frac{\Gamma(1,1,0)}{\Gamma(2,2,0)}$

Cn Defn.

$I(\vec r) = \Gamma(\vec r, \vec r, 0)$, $\quad S(\vec r, \omega) = \frac{1}{2\pi}\int \Gamma(r, r, \tau)e^{i\omega\tau}d\tau$

↳ INTENSITY $\qquad$ ↳ PWR SPEC. Wiener-Khinchin

Young's Expmt,

$E(\vec r, t) \propto \frac{1}{R_1}E\left(r_1, t - \frac{R_1}{c}\right) + \frac{1}{R_2}E\left(r_2, t - \frac{R_2}{c}\right)$

$I(r) \propto \frac{I(r_1)}{R_1} + \frac{I(r_2)}{R_2} + \frac{2}{R_1R_2}\mathrm{Re}\left\{\Gamma\left(1, 2, \frac{R_2 - R_1}{c}\right)\right\}$

$= I_1(\vec r) + I_2(\vec r) + 2\sqrt{I_1 I_2}\,\mathrm{Re}\left(\gamma\left(1, 2, \frac{R_2 - R_1}{c}\right)\right)$

$\approx 2I(r)\left\{1 + |\gamma_{12}|\cos\left(\frac{\omega r d}{cz} + \alpha\right)\right\}$

$\gamma(r_1, r_2, \tau) = \frac{\Gamma(1, 2, \tau)}{\sqrt{\Gamma(1,1,0)\Gamma(2,2,0)}}$

Props $\in [0, 1]$

$\gamma_{1,1,0} = 1$ $\qquad \star\, \Gamma(\tau) = \Gamma(r_1, r_2, \tau)$ if pol. field

Coherence generated by prop. γ_{12} of d.s. after ... $= \tilde I_0\left(\frac{kd}{R}\right)$ → expt

$$\left\langle\left|\sum_{n\geq1}\mathcal{E}(\vec r_n, t_n)a_n\right|^2\right\rangle \geq 0 \quad (\text{non-negative definite fn})$$