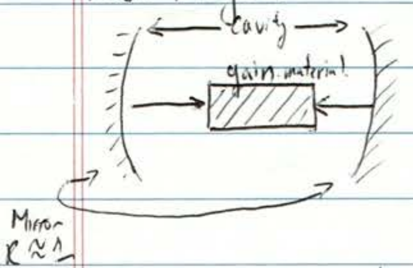
(PHY485)
Unit 1

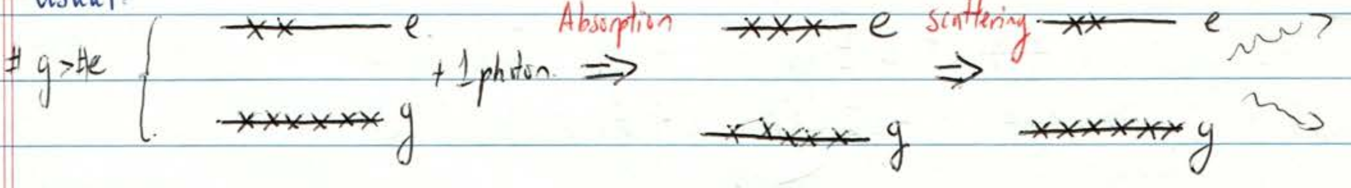
Laser Physics



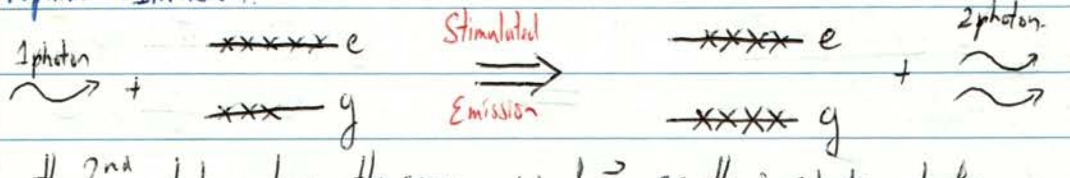
- Forms a standing wave
- gain material can be many things

What is stimulated emission?

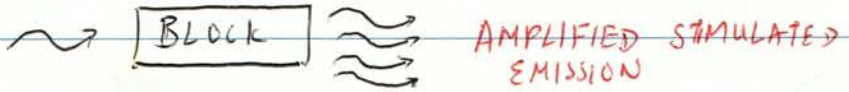
usual:



Population Inversion

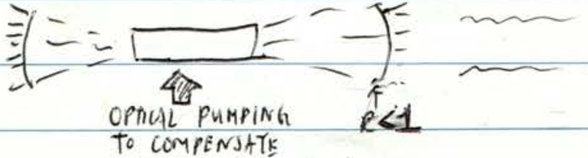


the 2nd photon has the same ω & $\vec{p}$ as the incident photon
↳ several fold in a block



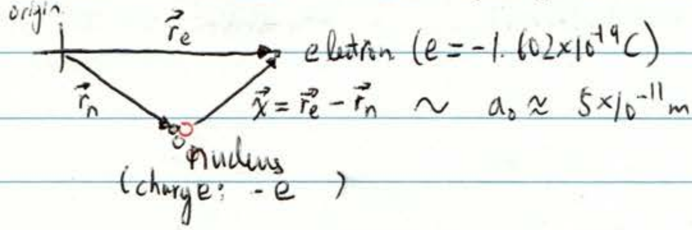
→ with a cavity, this is repeated.

EQUILIBRIUM
= % Abs
= % SE } equal



LASER BEAMS
- all @ same ω
- powerful! (directionality)
- coherent!
choir v.s. football match.

1. Absorption, Emission, & Dispersion of Light



no $\vec{B}$ term as $\vec{V}_e << c$
(non-relativistic)

Equations of Motions

$$m_e \frac{d^2 r_e}{dt^2} = e\vec{E}(\vec{r}_e, t) + \vec{F}_{e,n}(\vec{x}) \quad ①$$

$$m_n \frac{d^2 r_n}{dt^2} = -e\vec{E}(\vec{r}_n, t) + \vec{F}_{n,e}(\vec{x}) \quad ②$$

→ consider it constant for e & n, $\lambda \simeq 500\times10^{-9} >> a_0$

DIPOLE APPROXIMATION

CoM FoR

$$\vec{R} = \vec{r}_{com} = \frac{m_e\vec{r}_e + m_n\vec{r}_n}{m_e + m_n} \approx \vec{r}_n \qquad \Longrightarrow \qquad \vec{r}_e = \vec{R} + \frac{m_n}{M}\vec{x} \ ③$$

$$\vec{X} = \vec{r}_e - \vec{r}_n \qquad\qquad \vec{r}_n = \vec{R} - \frac{m_e}{M}\vec{x} \ ④$$

$M = m_n + m_e$

↳ reduced mass $m = \frac{m_e m_n}{M} \approx m_e$

Plug (3, 4) into 1, 2. (Taylor expansion near $\vec{R}$)

$$⑤ \quad m_e \frac{d^2R}{dt} + \frac{m_e m_n}{M}\frac{d^2\vec{x}}{dt^2} = e\vec{E}(\vec{R},t) + \frac{e\,m_n}{M}(\vec{x}\cdot\nabla_{\vec{R}})\vec{E} + \vec{F}_{en}$$

$$⑥ \quad m_n \frac{d^2R}{dt} + \frac{m_e m_n}{M}\frac{d^2\vec{x}}{dt^2} = -e\vec{E}(\vec{R},t) + \frac{e\,m_e}{M}(\vec{x}\cdot\nabla_{\vec{R}})\vec{E} - \vec{F}_{en}$$

⑤ + ⑥
(for beauty)

$$\boxed{M\frac{d^2R}{dt^2} = e(\vec{x}\cdot\nabla_{\vec{R}})\vec{E}(\vec{R},t)}$$

force on a dipole due to an external field.

- forms as the diff in the T.E.

& $m = \frac{m_e m_n}{M}$ (Harmonic mass)

By cancelling the $\frac{d^2R}{dt^2}$ term $\left(⑤ \times \frac{1}{m_e} - ⑥ \times \frac{1}{m_n}\right)$

$$⑦ \quad \overset{\frac{1}{m_j}}{\cancel{m\left(\frac{1}{m_e}+\frac{1}{m_n}\right)}} \frac{d^2x}{dt^2} = \frac{e}{m_e}\left\{\vec{E}(\vec{R}) + \frac{m_n}{M}(\vec{x}\cdot\nabla_{\vec{R}})\vec{E}(\vec{R})\right\} + \frac{e}{m}\left\{\vec{E}(\vec{R}) - \frac{m_e}{M}(\vec{x}\cdot\nabla_{\vec{R}})\vec{E}(\vec{R})\right\} + \overset{\frac{1}{m}}{\cancel{\left(\frac{1}{m_e}+\frac{1}{m_n}\right)}}\vec{F}_{en}$$

Using the definition of a harmonic mass, as well as $\nabla_R \vec{E} << \vec{E}$ (~~no~~ little variation of $\vec{E}$ in space w), we simplify to

$$⑧ \quad \boxed{\frac{d^2x}{dt^2} = \frac{e}{m}\vec{E}(\vec{R},t) + \frac{1}{m}\vec{F}_{en}}$$

effect of laser / external field on atomic spacing

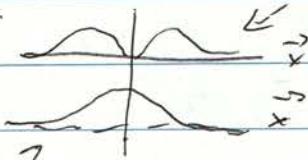
Potential Energy of the atomic system:

$$m\frac{d^2\vec{x}}{dt^2} = -\nabla_{\vec{x}} V \quad \vec{V} = -e\vec{x}\cdot\vec{E} = -\vec{d}\cdot\vec{E}$$

(dipole Hamiltonian)

We can have a simple SHO model for this

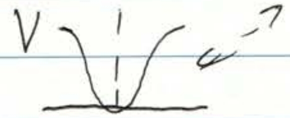
Limitations $|\Psi|^2$



$\vec{x}$ wrong near nucleus

$\vec{x}$ true.

Also



$$V(x) = V_0 + \frac{x^2}{2}k + \frac{x^2}{24}k'' + \dots$$

→ This solution is valid near equillibrium & for small perturbation

Modelling $F_{en} = -k\vec{x}$ & $\frac{k}{m} = w_0^2$ we get from ⑧

$$⑨ \quad \frac{d^2\vec{x}}{dt^2} + w_0^2\vec{x} = \frac{e}{m}\vec{E}(\vec{r},t) \; : \; \text{an SHO!!}$$

~~Power radiated by a dipole: (Larmour's Formula)~~

1.2 Spontaneous Decay

ignoring vector natur of E & x

$$⑩ \quad \ddot{x}(t) + w_0^2 x(t) = \frac{e}{m} E(0,t)$$

Given E defind as: $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E(\vec{r},t) = a\,x(t)\,\delta^{(3)}(\vec{r})$ (point sourc, dipole)

The dipole interacts with it self!!

☆ we get $⑪ \quad E(\vec{r},t) = \frac{-a}{4\pi}\frac{x(t-r/c)}{r} + E_0(\vec{r},t)$ {green's Fn Sol^n}

⑪ ⇒ ⑩, $\lim_{r\to 0}$

↳ $\lim_{r\to 0} E(0,t) = -\frac{a}{4\pi}\left\{\frac{x(t)}{r} - \frac{r}{c}\frac{\dot{x}(t)}{r} + \dots\right\} + E_0(0,t)$

$$⑫ \quad \ddot{x}(t) + w_0^2 x(t) = \frac{e}{m}E_0(0,t) \underbrace{- \frac{ae}{4\pi m}\left(\lim_{r\to 0}\frac{1}{r}\right)}_{\text{frequency shift}\to\ \delta\omega} x(t) + \underbrace{\frac{ae}{4\pi mc}}_{\gamma \leftarrow \text{decay}}\dot{x}(t)$$

☆ Lamb Shift

$$⑬ \quad \boxed{\ddot{x}(t) + \gamma\dot{x}(t) + w^2 x(t) = \frac{e}{m}E_0(0,t)}$$

baby's first model

→ extend to vector

Sol^n ⇒ $\vec{x}(t) = \vec{A}e^{-\gamma/2}\cos(w_1 t + \phi) \quad \left(w_1^2 = w_0^2 - \left(\frac{\gamma}{2}\right)^2\right)$

$$⑬b \quad \frac{d\vec{x}}{dt} = -w_1\vec{A}e^{-\frac{\gamma t}{2}}\sin(w_1 t + \phi) - \frac{\gamma}{2}e^{-\gamma/2}\vec{A}\cos(w_1 t + \phi)$$

Energy (T.A.) ⇒ $\mathcal{E} = \frac{m}{2}\left(\dot{x}^2 + w_0^2 x^2\right)$

$$⑭ \quad \cong \frac{mA^2}{2}e^{-\gamma t} \Rightarrow \text{loss over time.}$$

Power radiated by a dipole (Larmour's Formula)

$$⑮ \quad P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c^2}(\ddot{x})^2$$

$$= -\frac{\delta\mathcal{E}}{\delta t} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c^2}(\ddot{x})^2$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{2}{3}\right)\left(\frac{e^2}{c^3}\right)w_0^2 A^2 e^{-\gamma t}\cos^2(wt+\phi)$$

sub ⑬b

$$15a \quad \langle p\rangle = \underbrace{\frac{1}{4\pi\varepsilon_0}\left(\frac{2}{3}\right)\left(\frac{e^2}{c^3}\right)\left(\frac{w_1^2}{m}\right)}_{\gamma\ ?}\mathcal{E}$$

WRONG!

Full form of $\gamma = A_{21} := \frac{1}{4\pi\varepsilon_0}\frac{2e^2w_0^2}{mc^3}f$, where $f = \frac{2mw_0}{3\hbar}\langle x_{12}\rangle^2$

↳ oscillator strength

Derivation. $\nabla \vec{E} = \rho/\varepsilon_0 \qquad \nabla \times \vec{B} - \frac{1}{c^2}\frac{\delta}{\delta t}\vec{E} = \mu_0 \vec{J}$

$\nabla \vec{B} = 0 \quad \text{———} \quad \nabla \times \vec{E} + \frac{\delta B}{\delta t} = 0$ — — — MAXWELL

$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = -\frac{\delta \vec{A}}{\partial t} - \vec{\nabla}\varphi$

With Math: $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}', t - R/c)}{R}\, d^3 r' \qquad R = |r - r'|$

To get intensity in the far field, we need to integrate the Pointing Vector.

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$

$$\vec{B} = \nabla \times \hat{A} = \frac{1}{c}(\vec{A} \times \vec{n}) \qquad \vec{E} = ((\vec{A} \times \vec{n}) \times \vec{n})$$

in the dipole approximation: $\int \frac{J(r', t - |\vec{r} - \vec{r}'|/c)}{|r - r'|} = \frac{1}{r}\vec{d}$

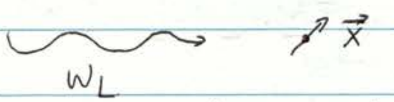
$$\vec{B} = \frac{\mu_0}{4\pi r c}\ddot{\vec{d}} \times n \qquad \vec{E} = \frac{\mu_0}{4\pi r}(\ddot{\vec{d}} \times \vec{n}) \times \vec{n}$$

$$\vec{S} = \frac{\mu_0}{16\pi^2 r^2 c}\vec{n}\,|\ddot{\vec{d}}|^2 \sin^2\theta$$, where θ is $\measuredangle\ \vec{d}$ & $\vec{n}$

$$P = \int_{4\pi} r^2 \vec{S} \cdot d\Omega = \frac{\mu_0}{16\pi^2 c} 2\pi |\ddot{\vec{d}}|^2 \underbrace{\int_0^\pi \sin^3\theta\, d\theta}_{4/3}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2}{3} e^2 \frac{\ddot{x}^2}{c^3}$$ ▨.

1.4 Absorption of Light



⑯ $$\left[\frac{d^2\vec{x}}{dt^2} + \gamma\frac{d\vec{x}}{dt} + \omega_0^2\vec{x} = \vec{\varepsilon}\,\frac{e}{m}E_0\cos(\omega_L + \phi)\right]$$

EOM

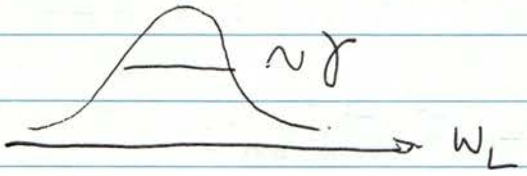
Solving it is hard, so FT!

⑰ $$\left(-\omega^2 - i\omega\gamma + \omega_0^2\right)\vec{X}(\omega) = \vec{\varepsilon}\,\frac{e}{m}\frac{E_0}{2}\left\{e^{i\phi}\delta(\omega+\omega_L) + e^{-i\phi}\delta(\omega-\omega_L)\right\}$$

$$X(e) = \vec{\varepsilon}\,\frac{e}{m}\frac{E_0}{2}\left\{e^{i\phi}\int_{-\infty}^{\infty}\frac{\delta(\omega+\omega_L)e^{i\omega t}}{-\omega^2 - i\omega\gamma + \omega_0^2}d\omega + e^{-i\phi}\int_{-\infty}^{\infty}\frac{\delta(\omega-\omega_L)e^{-i\omega t}}{-\omega^2 - i\omega\gamma + \omega_0^2}d\omega\right\}$$

⑱ $$\boxed{= \vec{\varepsilon}\,\frac{e}{m}E_0\,\mathrm{Re}\left\{\frac{e^{-i(\omega_L t + \phi)}}{-\omega_L^2 - i\omega_L\gamma + \omega_0^2}\right\}}$$

X(e), absorption spectra



Quick Aside: the Plank Spectrum $\rho(\nu) = \frac{8\pi h\nu^2/c^3}{\exp(\frac{-h\nu}{k_B T}) - 1}$ ⑲

Power losses by the field:

$$\frac{dE_{Field}}{dt} = \vec{F}\cdot\vec{V}$$
$$= \vec{\varepsilon}\,eE_0\cos(\omega_L t + \phi)\frac{d\vec{x}}{dt}$$

but $$\frac{d\vec{x}}{dt} = \vec{\varepsilon}\,\frac{eE_0}{m}\mathrm{Re}\left\{\frac{-i\omega_L e^{-i(\omega_L t + \phi)}}{-\omega_L^2 - i\omega_L\gamma + \omega_0^2}\right\}\mathrm{Re}\left\{\frac{-i\omega_L\{-\omega_L^2 + i\omega_L\gamma + \omega_0^2\}e^{-i(\omega_L t + \phi)}}{(\omega_0^2 - \omega_L^2)^2 - \omega_L^2\gamma^2}\right\}$$

FW.

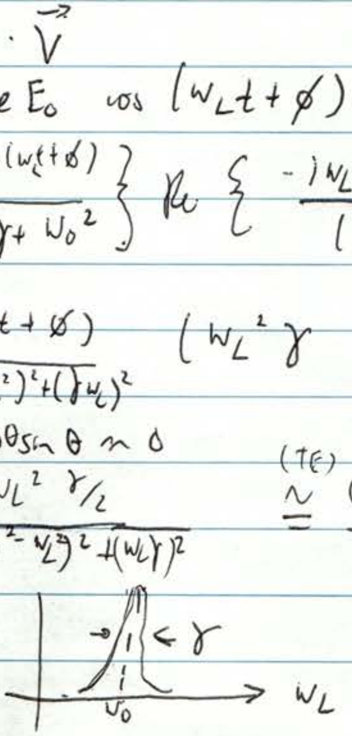
simplifying $$\frac{d\mathcal{E}_{Field}}{dt} = \left(\frac{eE_0}{m}\right)^2\frac{\cos(\omega_L t + \phi)}{(\omega_0^2 - \omega_L^2)^2 + (\gamma\omega_L)^2}\left(\omega_L^2\gamma\cos(\omega_L t + \phi) - \omega_L(\omega_0^2 - \omega_L^2)\sin(\sim\right.$$

TA: $\cos^2\theta \approx \frac{1}{2}$, $\cos\theta\sin\theta \sim 0$

⑳ $$\frac{d\mathcal{E}}{dt} = \frac{(eE_0)^2}{m}\frac{\omega_L^2\,\gamma/2}{(\omega_0^2 - \omega_L^2)^2 + (\omega_L\gamma)^2} \overset{(TE)}{\simeq} \frac{(eE_0)^2}{m}\frac{\omega_L^2\frac{\gamma}{2}}{4(\omega_0 - \omega_L)^2 + \omega_L^2\gamma^2}$$

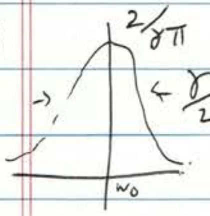
Absorption spectrum

$\omega_0 = \pm\omega_L$ peaks.



substitute $I = \frac{1}{2} c \varepsilon_0 E_0^2$ $\qquad \gamma = \omega_L / 2\pi$.

(21) $L(\omega_L)$



$$\frac{\partial E}{\partial t} = \frac{e^2}{4 m c \varepsilon_0} I \left\{ \frac{\gamma/2\pi}{(\omega_0-\omega_L)^2+(\gamma/2)^2} \right\}$$

RADIATIVE & BROADENING

$S(\nu)$ ↳ Lorentzian, can be replaced w/ $S(\nu)$

Gamma for ensemble, erf for brownian

- - - - - —

Generalization:

↳ Radiative broadening & collision are not the only broadening

{ } ⇒ goes to $S(\nu)$

↳ We'll fudge it for now, w/ osc strength, discrete levels, & SHO

(23) $$\frac{d\mathcal{E}}{dt} = -\frac{d}{dt}\left(\hbar \omega_0 N_1\right)$$ ← energy in level 1

rate eqn

equating 22 & 23

$$\frac{dN_1}{dt} = -\frac{1}{4\pi\varepsilon_0} \frac{\pi e^2 f}{m c \hbar \omega} N_1 I S(\nu)$$

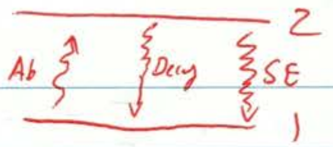
(24) (rate change due to ~~decay~~ Absorption) $$= -\frac{1}{h\nu} \frac{\lambda^2}{8\pi} A_{21} N_1 I S(\nu) = -\frac{\delta N_2}{dt}$$

↑ single band.

For Broad band: $\left(S(\nu) \text{ for } \nu_1, \nu_2\ \nu_3\right)$

$$\frac{dN_1}{dt} = \frac{-A_{21} N_1}{8\pi h} \int_0^\infty \frac{c^2}{\nu^3} I(\nu) S(\nu)\, \delta\nu$$ $\qquad I(\nu) = c\,\rho(\nu_0)$

(25) $$\approx \frac{-A_{21} N_1 c^3}{8\pi h \nu_0^3} \rho(\nu_0)$$



A&B coefficients & Thermal Equilibrium.

Things affecting the Rate Eq^n: absorption & spontaneous decay & STIMULATED Emission

Wien only gets the first two ~~Pl~~ Planck modifies it for you.

$$\hookrightarrow \frac{dN_1}{dt} = A_{21} N_2 - \frac{A_{21}}{8\pi h} \frac{c^3}{\nu_0^3} N_1 \rho(\nu_0) = 0$$

in Equilibrium, $\frac{N_2}{N_1} = e^{-\frac{h\nu_0}{k_B T}}$ Boltzman factors ⓐ(26a)

$$"\rho"(\nu_0) = \frac{8\pi h \nu_0^3}{c^3} e^{-h\nu_0/k_B T} \Rightarrow \text{Wien spectrum} : \text{WRONG!}$$

(26) Planck: Rate Eq^n

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt} = \underbrace{A_{21} N_2}_{\text{DECAY}} - \underbrace{\frac{A_{21}}{8\pi h} \frac{c^3}{\nu_0^3} N_1 \rho(\nu_0)}_{\text{ABSORPTION}} + \underbrace{\frac{A_{21} c^3}{8\pi h \nu_0^3} N_2 \rho(\nu_0)}_{\text{STIMULATED EMISSION}}$$

Absorption Cross Section

$$\frac{dN_1}{dt} = -\frac{1}{h\nu} \frac{\lambda^2}{8\pi} A_{21} N_1 \overset{c\rho(\nu_0)}{I} \underset{\text{normalized} : \int S(\nu) d\nu = 1}{S(\nu)} :$$

Absorp cross section $\boxed{\sigma(\nu) = \frac{\lambda^2}{8\pi} A_{21} S(\nu)}$ (27)

$$\frac{dN_1}{dt} = -\frac{\sigma(\nu)}{h\nu} I N_1$$

Notes:

$S(\nu_0) = \pi / \nu_1 \; ?? = \frac{4\pi^2}{A_{21}}$ @ ν_0, atom has effective area $\propto \lambda^2$

$\sigma(\nu_0) = \frac{\pi}{2} \lambda^2$: real area is ~ A: $\propto a_0^2$

(28) Two level. Thus

$$\boxed{\frac{dN_2}{dt} = -\frac{dN_1}{dt} = -\frac{\sigma(\nu)}{h\nu} (N_2 - N_1) I - A_{21} N_2}$$

Two Wrinkles: degeneracy & Exotic types o broadeny.

Degeneracy

Ideal

— N_2

— N_1

Real!

$m_2 = \underline{-J_2}\quad \underline{-J_2+1}\quad \_\_\quad \_\quad \_\quad \underline{J_2}$ $\quad \# = g_2$

$m_1 = \underline{-J_1}\quad \_\_\quad \_\quad \_\quad \underline{J_1}$ $\quad \# = g_1$

New RWA $$\frac{d\mathcal{N}_2(m_2)}{dt} = -\sum_{m_1}\left\{\mathcal{R}(m_2,m_1)\left(\mathcal{N}_2(m_2) - \mathcal{N}_1(m_1)\right) + \mathcal{A}(m_2,m_1)\mathcal{N}_2(m_2)\right\}$$

$$\frac{dN_2}{dt} = \sum_{m_2}\frac{d\mathcal{N}_2(m_2)}{dt} = \sum_{m_2}\{\quad\downarrow\quad\} \;\hat{=}\; ㉙$$

Assume uniform distribution: $\mathcal{N}_2(m_2) = \frac{N_2}{g_2}$ & $\mathcal{N}_1(m_1) = \frac{N_1}{g_1}$ ㉚ a, b.

From eqn (28-30 a, b), we get:

$$A_{21} = \frac{1}{g_2}\sum_{m_1 m_2}\mathcal{A}(m_2,m_1)$$

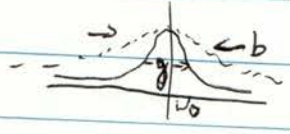
$$\left.\begin{aligned}\frac{1}{c}IS(\nu)B_{21} &= \frac{1}{g_2}\sum_{m_1 m_2}\mathcal{R}(m_1,m_2)\\ \frac{1}{c}IS(\nu)B_{12} &= \frac{1}{g_1}\sum_{m_1 m_2}\mathcal{R}(m_1,m_2)\end{aligned}\right\}\ \frac{g_2}{g_1}B_{21} = B_{12}$$

This gives us a rate eqn of

㉛ $$\frac{dN_2}{dt} = -\frac{1}{h\nu}\sigma(\nu)I\left(N_2 - \frac{g_2}{g_1}N_1\right) - A_{21}N_2$$

Broadening Pt 2 : Electric Boogaloo

i) Collision Broadening : approximated by modif. $\beta = \gamma + \sim$; many assumptions made



$$\frac{\partial^2 \vec{x}}{\partial t^2} + \beta \frac{\partial \vec{x}}{\partial t} + \omega_0^2 \vec{x} = \frac{-e}{m} \vec{E}(0,t)$$

☆ This is a form of homogenous broadening, affecting all atoms equally

ii) Doppler broadening

$\nu = \nu_0 = -\nu_0 \frac{u}{c}$

a Maxwell Boltzmann distribution ~~results~~ shows us avg speed.

speed → $p(u)\,du$ = Probability that atom ∈ $(u, u+du)$

$$= \sqrt{\frac{M}{2\pi k_B T}}\, e^{-\frac{Mu^2}{2k_B T}}$$

in absence of collisional Broadening:

$$\boxed{S(\nu) = \frac{1}{\sqrt{\pi}(\delta\nu_D)} \exp\left(-\frac{(\nu-\nu_0)^2}{2(\delta\nu_D)^2}\right)}$$ 8ln

doppler Width $\delta\nu_D = \sqrt{\frac{k_B T}{M}} \frac{\nu_0}{c}$

Lorentzian distribution ↳ wider than gaussian

N.B: there's sometimes a 2 or a ln2 added for σ, HWHH, etc.

☆ This is a form of inhomogenous broadening, affecting each atom differently

iii) both: $S_{eff}(\nu) = \int_{-\infty}^{\infty} S(\nu, u) \sqrt{\frac{M}{2\pi k_B T}} \exp\left(-\frac{Mu^2}{2k_B T}\right) du.$

$$= \int_{-\infty}^{\infty} \frac{\partial\nu/\pi}{(\nu_0 - \nu + \nu_0 \frac{u}{c})^2 + \delta\nu_0^2} \sqrt{\frac{M}{2k_B T}} \frac{\cancel{c}}{\cancel{\nu_0}} \exp\left(\frac{-Mu^2}{2k_B T}\right) du.$$

⇒ with $y = \sqrt{\frac{M}{2k_B T}}\, u.$ $x = \frac{(\nu_0 - \nu)}{\delta\nu}$ ⇒ $b = \sqrt{\frac{M}{2k_B}} \frac{c\,\partial\nu_0}{\nu_0}$

VOIGHT PROFILE $S_{eff}(\nu) = \frac{1}{\pi^{3/2}} \frac{b^2}{\delta\nu_0} \int_{-\infty}^{\infty} \frac{\delta y\, e^{-y^2}}{(y-x)^2 + b^2}$

1.6 Refractive index (derivation from first principles)

MAXWELLS

$$\nabla\cdot\vec{D} = \rho f \quad (1a) \qquad \nabla\times\vec{H} - \frac{d\vec{D}}{dt} = \vec{J}f \quad (1b)$$

$$\nabla\cdot\vec{B} = 0 \quad (2a) \qquad \vec{\nabla}\times\vec{E} + \frac{d\vec{B}}{dt} = 0 \quad (2b)$$

Assume $\rho f = Jf = \vec{M} = 0$

→ Modify Eq(1): $\vec{\nabla}\cdot E = -\frac{1}{\varepsilon_0}\vec{\nabla}\cdot\vec{P}$ (1'a) (split up $\vec{D}$)

$$\vec{\nabla}\times\vec{B} = \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} + \mu_0 \frac{d\vec{P}}{dt} \quad (1'b)$$

↳ This comes from the linear response of material ⇒ Use it to express the eqn purely in terms of $\vec{E}$

~~(29a)~~ ㉜ $P(t) = \epsilon_0 \chi \vec{E}(t)$ (eq 3) ~~③~~

$$= \epsilon_0 \int_{-\infty}^{\infty} \chi(t-t')\vec{E}(t')\,dt',$$

→ allows for resonance.

→ $\chi(t>0) = 0$ (no future, causality).

Take the curl of 2b, and substitute 1'b into it

$$\nabla\times\nabla\times\vec{E} + \left(\frac{d}{dt}(\nabla\times\vec{B})\right) = 0$$

$$-\nabla^2\vec{E} + \underset{\substack{\text{in FF field,}\\ =0,\ \text{fields transverse}}}{\nabla(\nabla\cdot\vec{E})} + \mu_0\epsilon_0 \frac{d^2E}{dt^2} + \mu_0\frac{d^2\vec{P}}{dt^2} = 0$$

Substituting in (3)

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{d^2}{dt^2}\int_{-\infty}^{\infty}\overset{\text{convolution.}}{\left\{\vec{\chi}(t-t')E(t')dt'\right\}} = 0 \quad (4)$$

~~(30)~~ ㉝ FT. $\nabla^2\tilde{E}(\omega) - \frac{1}{c^2}(i\omega)^2 2\pi\tilde{\chi}(\omega)\ \tilde{E}(\omega) = 0$ (5)

HELMHOLTZ. $\left[\nabla^2 + \frac{\omega^2}{c^2}(1+2\pi\hat{\chi}(\omega))\right]\tilde{E}(\omega) = 0$

~~(31)~~ ㉞ ↳ $n(\omega) = n_1(\omega) + i n_2(\omega) = \sqrt{1+2\pi\tilde{\chi}(\omega)}$

H Heqn can be solved as a monochromatic, plane wave.

$$\because \vec{E}(t) \in \mathbb{R}., \quad \tilde{\vec{E}}(w) = \tilde{E}(-w)^*$$

u ∈ direct vector, unit length

const, ∈ℝ.

$$\tilde{E}(w) = \frac{\mathcal{E}_0}{2}\{\delta(w-w_0)\exp\left(\frac{iw}{c}n(w_0)\vec{u}\cdot\vec{r}\right) + \delta(w+w_0)\exp\left[\frac{-iw_0 n^*(w_0)}{c}\vec{u}\cdot\vec{r}\right]$$

★ $\vec{\mathcal{E}}_0 \cdot \vec{u} = 0$ (polarization is transverse) $\quad \vec{u}\cdot\vec{u} = 0$

㉟ $\boxed{\text{IFT}}$

$$E(t) = \frac{\vec{\mathcal{E}}_0}{2}\left(\exp\left[\frac{iw_0}{c}n(w_0)\vec{u}\cdot\vec{r}\right] * \exp[-w_0 t] + \exp\left[\frac{-iw_0}{c}n(w_0)\vec{u}\cdot\vec{r}\right] * \exp[w_0 t]\right)$$

$$E(t) = \vec{\mathcal{E}}_0 \cos\left(\left[\frac{w_0}{c}n_1(w_0)\left[\vec{u}\vec{r} - \frac{c}{n_1(w_0)}t\right]\right]\right)\exp\left(-\frac{\beta}{2}\vec{u}\cdot t\right)$$

where $\beta = \frac{2w_0}{c}n_2(w_0)$ = decay (absorption)

★ $V_{\phi} = \frac{c}{n_1(w_0)}$
phase

Kramers-Kröning Relation.

$$\chi(t) = \chi(t)\,\textcircled{H}(t) \quad : \quad \textcircled{H}(t) = u[t]$$

$$\hat{\chi}(w) = \int_{-\infty}^{\infty} \tilde{\chi}(w')\,\hat{\textcircled{H}}(w-w')\,dw'$$

★ $$\tilde{\textcircled{H}}(w) = \frac{1}{2}\delta(w) - \frac{1}{2\pi i w}$$

$$= -\frac{1}{2\pi}\zeta(w) \quad \text{(Heitler Zeta fn)}$$

$$\hat{\chi}(w) = \frac{1}{2}\hat{\chi}(w) - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\chi}(w')}{(w-w')}\,dw$$

$$= \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\chi}(w')}{w'-w}\,dw'$$

but $\chi(w) = \chi_1(w) + i\chi_2(w)$
$\Rightarrow$

$$\left.\begin{aligned}\tilde{\chi}_1(w) &= \frac{1}{\pi}\int_{\infty}^{\infty}\frac{\tilde{\chi}_2(w')}{w'-w}\,dw' \\ \chi_2(w) &= -\frac{1}{\pi}\int_{\infty}^{\infty}\frac{\tilde{\chi}_1(w)}{w'-w}\,dw'\end{aligned}\right\}$$ KKR

㉝.

2: Laser Oscillations

2.1 GAIN & THRESHOLDING = Change in beam as a function of t & z

$$\underset{\text{Poynting Vector}}{\nabla\cdot\vec{S}} = \frac{\partial (u)}{\partial t} \leftarrow \text{E/m Field Energy Density}$$

$\frac{g_2}{g_1} B_{21} = B_{12}$

For beam: $\vec{S} = \hat{e}_z I = \hat{e}_z uc$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) I = \underset{\text{SE cross section}}{\sigma}\left(\underset{\substack{\text{density of}\\\text{level 2}\\\text{(SE)}}}{N_2} - \frac{g_2}{g_1}\underset{\substack{\text{density of}\\\text{lower level}\\\text{(Absorption)}}}{N_1}\right) I$$

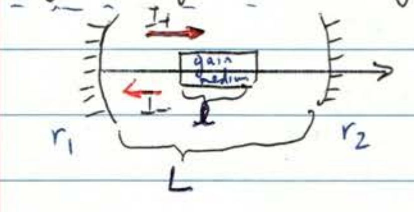
㉜ gain coefficient: $g = \sigma\left(N_2 - \frac{g_2}{g_1} N_1\right)$

↳ No decay accounted for due to emission in all Steradians v-s $\hat{z}$ is negligible

For the steady state: $\frac{dI}{dt} = 0$: $\frac{d}{dz} I = gI$

㉝ $I = I(0)\, e^{gz}$ ★ (small signal case, no saturation / depletion of $\frac{N_1}{N_2}$)

$g > 0$ gain $\quad g < 0$ attenuation



Typical Laser Conditions

(B.C) @ $z=0$ $\quad I_+(0) = r_1 I_-(0)$ [BC1]

@ $z=L$ $\quad I_-(L) = r_2 I_+(L)$ [BC2]

$$\begin{aligned} I_-(0) &= I_-(L)\, e^{gl} \\ &= r_2 I_+(L) e^{gl} \quad (BC2) \\ &= r_2 I_+(0) e^{2gl} \\ &= r_1 r_2 I_+(0) e^{2gl} \quad (BC1) \end{aligned}$$

$\Rightarrow$ for SS operation: $r_1 r_2 e^{2gl} = \underline{1}$

solve for g : $g_{th} = -\frac{1}{2\ell}\ln(r_1 r_2)$ ★ threshold gain?

(34) $$\boxed{g_{th} \approx \frac{1-r_1 r_2}{2\ell}}$$

$r_1 r_2$ → reflectivity (for I)

2ℓ → length of gain medium.

↳ this is the typical experimental gain in SS operation (due to various effects)

2.2 Two level System Rate Equation

$\frac{I}{h\nu} = \Phi$

$$\dot{N}_1 = A_{21} N_2 + g\Phi$$
$$\dot{N}_2 = \underbrace{-A_{21} N_2}_{\text{decay}} - \underbrace{g\Phi}_{\text{SE/Absorption}}$$
$$\dot{\Phi} = \frac{c\ell g}{L}\Phi - \frac{c}{2L}(1-r_2 r_1)\Phi$$

initial rate eq^n's

ℓ/L × cavity : gain on photons

$\frac{c}{2L}(1-r_2r_1)\Phi$ → lost through mirrors

Assume Φ const for now ie. $N_1 + N_2 = N$ const, bad assumption

$$\dot{N}_2 = -A_{21} N_2 - \sigma(2N_2 - N)\Phi$$
$$= -(A_{21} + 2\sigma\Phi)N_2 + \sigma\Phi N$$

$$\dot{N}_2 + \alpha N_2 = \sigma\Phi N$$
$$e^{-\alpha t}\frac{d}{dt}(e^{\alpha t}N_2) \Rightarrow e^{\alpha t}N_2(t) - N_2(0)$$
$$= \int_t \sigma\Phi N e^{\alpha t}$$
$$= \frac{\sigma\Phi N}{\alpha}(e^{\alpha t} - 1)$$
$$N_2(t) = N_2(0)e^{-\alpha t} + \frac{\sigma\Phi N}{\alpha}(1 - e^{\alpha t})$$

(35) $$= \underbrace{\left[N_2(0) - \frac{\sigma\Phi N}{A_{21}+2\sigma\Phi}\right]e^{-(A_{21}+2\sigma\Phi)t}}_{\text{Transient} \to 0 \text{ as } t\to\infty} + \underbrace{\frac{N\sigma\Phi}{A_{21}+2\sigma\Phi}}_{\text{S.S. sol}^n}$$

$N_2(\infty) = \begin{cases} \text{Weak Excitation. } \sigma\Phi \ll A_{21} \Rightarrow N_2(\infty) \ll N \\ \hookrightarrow \text{most of } N \text{ in level } 1 \\ \text{Strong Excitation } \sigma\Phi \gg A_{21} \Rightarrow N_2(\infty) \approx \frac{1}{2}N \end{cases}$

↳ Saturation → $N_2 - N_1 = 0$, no gain! :(

- - - - - - - - -

Aside: Broadening during steady state operation:

(arbitrary N) $$N_2(\infty) \approx \frac{N\sigma\Phi}{A_{21} + 2\sigma\Phi} \quad ; \quad \sigma = \frac{\sigma_0 \beta/\pi}{\beta^2 + (w-w_0)^2}$$

$$= \frac{N \sigma_0 \beta/\pi \, \Phi}{A_{21}(\beta^2 + (w-w_0)^2) + 2\sigma_0 \beta/\pi \, \Phi}$$

$$= \frac{N \sigma_0 \beta \, \Phi/(\pi A_{21})}{(w-w_0)^2 + \beta^2 + \underbrace{\frac{2\sigma_0\beta\Phi}{\pi A_{21}}}_{\text{Broadening terms due to } \Phi}}$$

- - - - - - - - -

2.3 3 level systems & Pumping:

Formulas from 2.2 that are necessary:

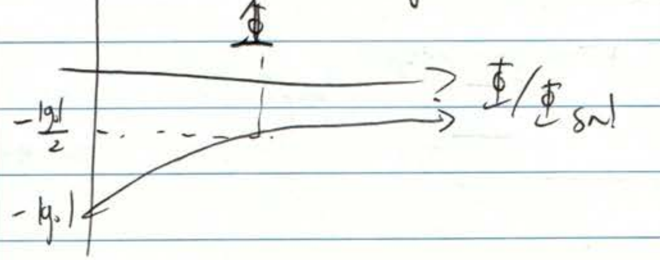
steady state: $$N_2 = \frac{N\sigma\Phi}{A_{21} + 2\sigma\Phi} \qquad N_1 = \frac{N(A_{21} + \sigma\Phi)}{A_{21} + 2\sigma\Phi}$$ (36)

$$g = \sigma(N_2 - N_1) = -\frac{N A_{21} \sigma}{A_{21} + 2\sigma\Phi}$$

Small signal gain: $g(A_2 \gg \sigma\Phi) = -N\sigma$

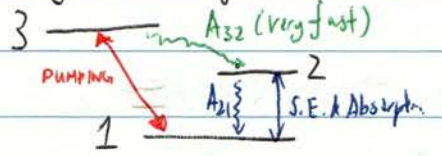
Saturation flux: $\Phi_{sat} = A_{21}/2\sigma$

$$g = \frac{g_0}{1 + \Phi/\Phi_{sat}}$$ (37)



2 level will never have $g > 0$ (gain)

3 level system analysis:



rate P

A_{32} is BIG., so $N_3 \simeq 0$

$$\left.\begin{aligned}\dot{N}_1 &= A_{21}N_2 + \sigma(N_2-N_1)\Phi + P(N_3-N_1)\\ \dot{N}_2 &= -A_{21}N_2 - \sigma(N_2-N_1)\Phi + A_{32}N_3\\ \dot{N}_3 &= -P(N_3-N_1) - A_{32}N_3\end{aligned}\right\}(RE\ 1,2,3)$$

Assume SS $(\dot{N}_1 = \dot{N}_2 = \dot{N}_3 = 0)$

$N_3 = N_1 \frac{P}{A_{32}+P}$ ⇒ sub that into RE 1, 2.

$$\left.\begin{aligned}\dot{N}_1 &= A_{21}N_2 + \sigma(N_2-N_1)\Phi - \frac{PN_1A_{32}}{P+A_{32}}\\ \dot{N}_2 &= -A_{21}N_2 \mp \sigma(N_2-N_1)\Phi + A_{32}\frac{PN_1}{P+A_{32}}\end{aligned}\right\}$$ reclassify as $\gamma = \frac{A_{32}}{A_{32}+P}P$
$= P$ if $A_{32} >> 1$

↓

$$\dot{N}_1 = (A_{21}-\gamma)N_2 + (\sigma\Phi + \gamma)(N_2 - N_1)$$

Comparison to two level:

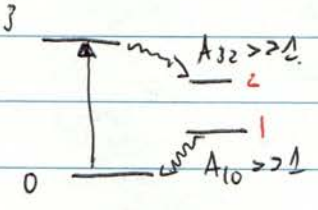
↳ $A_{21} \leftrightarrow A_{21} - \gamma$
$\sigma\Phi \longleftrightarrow \sigma\Phi + \gamma$

⋮

$$g = \frac{g_0}{1+\Phi/\Phi_{sat}}$$ where $$g = \frac{N(\gamma - A_{21})}{P + A_{21}}\sigma$$
$$\Phi_{sat} = \left(\frac{A_{21}+\gamma}{2\sigma}\right)$$

As long as $\gamma > A_{21}$, gain occurs!

Extensible to 4 level:



} pop^n inversion is easy
↳ gain almost guaranteed

let us consider gain as a fn of (ν) for a general system

$$g(\nu) = \frac{g_0(\nu)}{1 + \underset{\Phi(\nu)}{\Phi} / \Phi_{sat}} \quad \begin{matrix} \leftarrow \text{small signal gain} \\ \leftarrow \text{sat flux:} \end{matrix}$$

where $g_0 = \frac{N(P - A_{21})}{P + A_{21}} \sigma(\nu)$ $\quad \Phi_{sat} = \frac{P + A_{21}}{2\sigma(\nu)}$, $\sigma(\nu)$ resont @ ν_0

Resonant behaviour: $$g_0(\nu) = \frac{g_0(\nu_0)}{1 + \left(\frac{\nu - \nu_0}{\delta\nu_0}\right)^2}$$

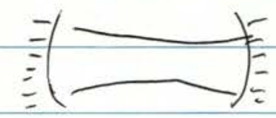
$$\Phi_{sat} = \Phi(\nu_0)\left\{1 + \left(\frac{\nu - \nu_0}{\delta\nu_0}\right)^2\right\}$$

$$g(\nu) = \frac{g(\nu_0)}{1 + \left(\frac{\nu - \nu_0}{\delta\nu_0}\right)^2 + \Phi/\Phi_{sat}}$$

→. This causes broadening: - line is widened if Φ is large.
- difficult to sat if $\nu \neq \nu_0$

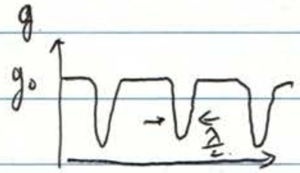
Ben has inhomogeneous & main identities

2.4. Spatial Hole Burning



$$E = 2E_0 \cos(\omega t)\sin(kz) \qquad \omega >> k$$

$$I = h\nu\Phi = \varepsilon_0 c E^2$$
$$= \varepsilon_0 4 E_0^2 \overset{\frac{1}{2}}{\cancel{\cos^2(\omega t)}} \sin^2(kz)$$
$$\approx 2\Phi_0 \sin^2(kz)$$



$$g = \frac{g_0(\nu)}{1 + \frac{2\Phi_0}{\Phi_{sat}}\sin^2(kz)}$$

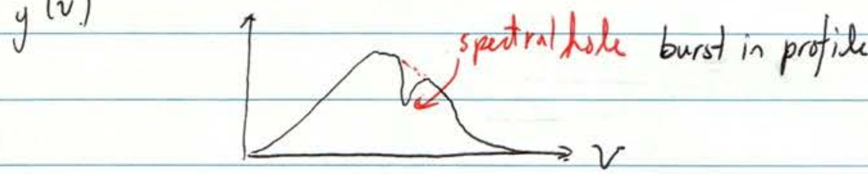
⇒ if you T.A. the spatial oscilations,

$$g = \frac{g_0(\nu)}{1 + I_0/I_{sat}} \text{ as usual.}$$

2.5 Spectral Hole Burning

→ occurs in a inhomogenously broadened medium
  → when diff atoms have diff resonance freqs
  → E.g. Doppler Broadening, Isotope Shifts, Zeeman / Stark shifts $\vec{B}$ / $\vec{E}$

→ Atoms that are on resonance w/ the laser mode will saturate more easily than those off resonance



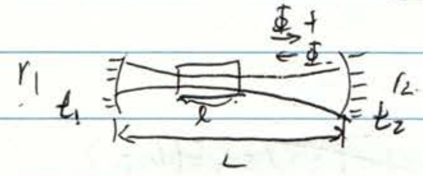
2.6. Gain Clamping & Output power

- homogenously broadening
- uniform field approximation
- continuous wave, CW i.e. S.S. operation

$$\hookrightarrow \frac{d}{dt}(\Phi, N_1, N_2) = 0 : \text{steady}$$

Rate of photon flux $\left(\frac{d\Phi}{dt}\right)$

$$\frac{d\Phi}{dt} = \underbrace{\frac{cl}{L} g(v)\ \Phi}_{\text{gain}} - \frac{\overset{\text{speed of light}}{c}}{2L}(1 - r_1 r_2)\Phi$$



if $\frac{d\Phi}{dt} \approx 0$ $\quad g(v) = \frac{1}{2l}(1 - r_1 r_2)$ (threshold gain)

$\llcorner$ gain is effectively "clamped"!

$$\boxed{\frac{g_0(v)}{g_{Th}} = 1 + \Phi/\Phi_{sat}}$$

$$\Phi = \Phi_{sat}\Big(\underbrace{\frac{g_0(v)}{g_{Th}}}_{\gg 1} - 1\Big)$$ Intra cavity ~~p~~ photon flux

$$\Phi_{out} = t_1 \Phi(-) + t_2 \Phi(+)$$

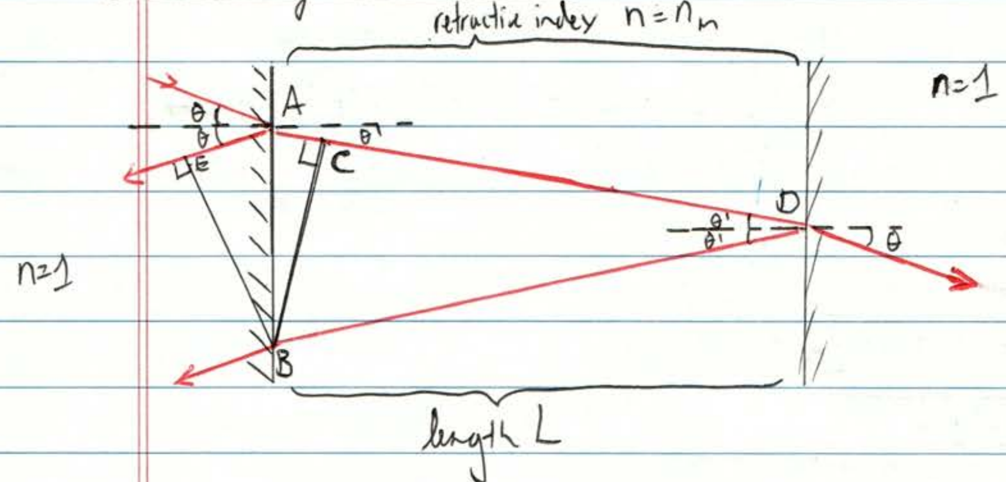
$$= \left(\frac{t_1 + t_2}{2}\right)\Phi$$

$$= \frac{t}{2}\ \Phi_{sat}\left(\frac{g_0(v)}{g_{Th}} - 1\right) = l\,\Phi_{sat}\,(g_0(v) - g_{Th})$$

$$I_{out} = \frac{l\,N(P - A_{21})}{2} \quad \text{when } I_{pump} = P(h\nu_{13})$$

PHY 485: Unit 3: Cavities and Resonators

§ 3.1 Fabry - Perot Etalon



snell's law

$n_1 \sin\theta_1 = n_2 \sin_2\theta$

Geometry

$$AE = AB\sin\theta$$
$$= ABn\sin\theta' = nAC$$
$$AD = BD = L/\cos(\theta')$$
$$DC = BD\cos(2\theta')$$
$$= \frac{L}{\cos(\theta')}\cos(2\theta')$$
$$= 2L\cos(\theta') - \frac{L}{\cos(\theta')}$$

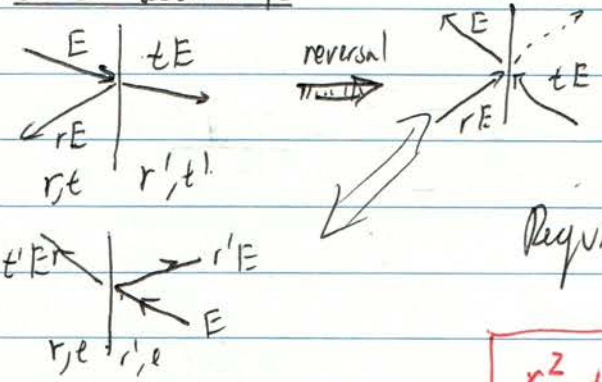
Phasedelay btwn adjacent rays

$$\delta = \frac{2\pi}{\lambda}\left(n(AD + BB) - AE\right)$$
$$= \frac{2\pi n}{\lambda}\left(AD + BB - AC\right)$$
$$= \frac{2\pi n}{\lambda}\left(DC + BD\right)$$
$$= \frac{2\pi n}{\lambda}\left(2L\cos(\theta') - \cancel{\frac{L}{\cos(\theta')}} + \cancel{\frac{L}{\cos(\theta')}}\right)$$
$$= 2\pi\left(\frac{2Ln\cos(\theta')}{c}\right)\nu$$

Define $\boxed{\Delta\nu = \frac{c}{2Ln\cos(\theta')}}$ as the Free Spectral Range

$$\delta = 2\pi\frac{\nu}{\Delta\nu}$$

Stokes Relations



Reversall should equate these two cases

Requires: $r(rE) + tt'E = E$

$t(rE) + r't'E = 0$

$$\boxed{\begin{array}{c} r^2 + tt' = 1 \\ t(r + r') = 0 \end{array}}$$

Stokes reciprocity relations

Back to the Etalon: what is the effective coefficient? et.

$$r_{eff} = r_1 + t_1 r_2' t_1' e^{i\delta} + t_1 r_2' r_1' r_2' t_1' e^{i2\delta} + t_1 r_2' r_1' r_2' r_1' r_2' t_1' e^{i3\delta} + \dots$$

$$\overset{\text{stokes}}{=} (-r_1') + r_2' t_1 t_1' e^{i\delta}\left(1 + r_1' r_2' e^{i\delta} + (r_1 r_2 e^{i\delta})^2 + \dots\right)$$

$$= -r_1' + r_2 t_1 t_1' e^{i\delta}\left(\frac{1}{1 - r_1' r_2' e^{i\delta}}\right)$$

$$\boxed{= \frac{-r_1' + \left[(r_1')^2 r_2' + r_2 t_1 t_1'\right] e^{i\delta}}{1 - r_1' r_2' e^{i\delta}}}$$

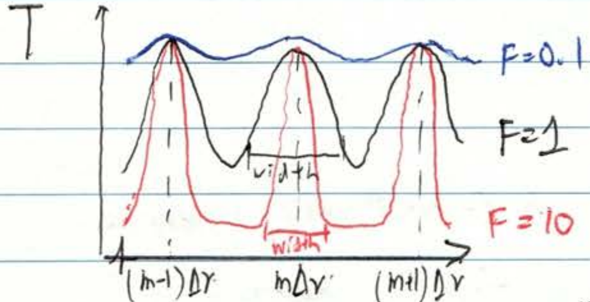
define $r_1' r_2' = R$

$$|r_e|^2 = \frac{4R\sin^2\left(\frac{\delta}{2}\right) + (r_1' - r_2')^2}{(1-R)^2 + 4R\sin^2\left(\frac{\delta}{2}\right)}$$

Other Parameters

$$T_e = 1 - |r_e|^2 = 1 - R_e$$

$$T_e = \frac{T_{max}}{1 + F\sin^2\left(\frac{\pi\nu}{\Delta\nu}\right)}$$ when $F = \frac{4R}{(1-R)^2}$ & $T_{max} = 1 - \left(\frac{r_1' - r_2'}{1 - r_1' r_2'}\right)$



Width of resonances

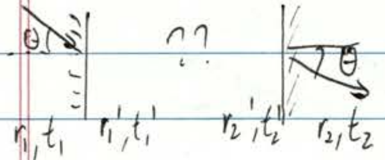
HWHM: $\delta\nu = \frac{\Delta\nu}{\pi\sqrt{F}}$

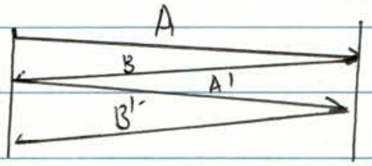
Finesse: $\mathcal{F} = \frac{\pi\sqrt{F}}{2} = \frac{\pi\sqrt{R}}{1-R}$

$= \sim 30 - 100$ is good

L11

Given an etalon, what happens btwn the mirrors?





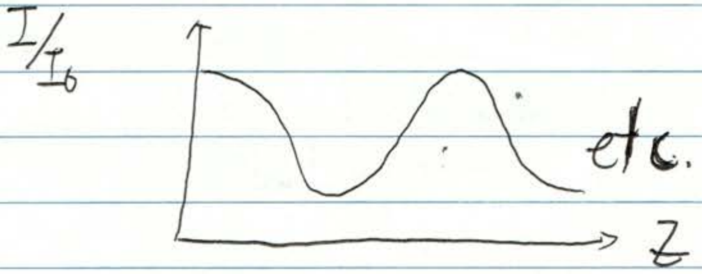
$$A: = E_0 \exp\left[ik\left(-(\sin\theta')x + (\cos\theta)z - ct\right)\right]$$
$$= E_0 \exp(ik\Phi^+)$$
$$B: = r_2' E_0 \exp\left[ik\left(-(\sin\theta')x - (\cos\theta')z - ct\right)\right]$$
$$= r_2' E_0 \exp(ik\Phi^-)$$

Thus: $$E_{0_{\text{inside}}} = E_0 \underbrace{\left\{e^{ik\Phi^+} + r_2' e^{ik\Phi^-}\right\}}_{1 - r_1' r_2' e^{i\delta}}$$

$$I(z) = I_0 \left\{\frac{1 - \frac{4r_2'}{(1+r_2')^2}\sin^2\left(\frac{\pi\nu}{\Delta\nu}\frac{z}{L}\right)}{1 + F\sin^2\left(\frac{\pi\nu}{\Delta\nu}\right)}\right\}$$

$r_2' = 1 - \xi$

$4r_2' = \frac{1-\xi^2}{4}$



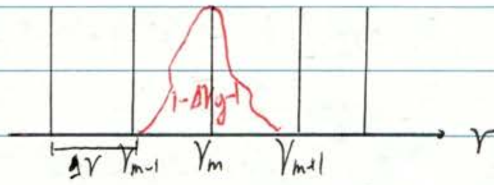
allowed longitudinal modes

$\nu_m = (\Delta\nu)m$, where $m \in \mathbb{Z}$

well, how can you limit it to just 1 mode?

§ 3.2 single mode operation (for CW, i.e., SS opp)

as mentioned, longitudinal modes: $\nu_m = m \Delta\nu$
$= m \frac{c}{2L}$ (if $\theta' = 0$) [all on z]



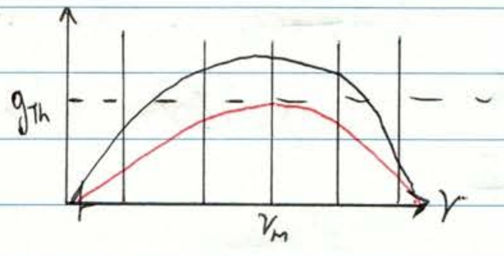
$\Delta\nu_g$ = width of gain

Option 1: have gain can be narrow, so it only interacts w/ 1 mode
if $\Delta\nu >> \Delta\nu_g$ then this is true: only 1 mode lases

$$\frac{c}{2L} >> \Delta\nu_g \Rightarrow L << \frac{c}{2\Delta\nu_g}$$

if we minimize L, g is minimized as well: low output :-(

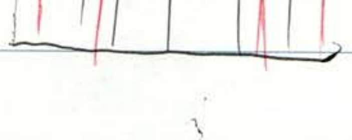
Option 2: gain clamping ⇒ mode with largest small signal gain saturates the gain: thus all other modes will not lase



Since only $g(\nu_m) = g_{Th}$, ν_m is the only mode that lases

Note: This is only guaranteed for homogeneous broadening:
Spatial hole burning can mess up line shapes

This can be made more robust w/ another tuned Faby Perot to the frequency & disallowing the other modes.
like



etc

§3.3 Laser Bandwidth

$$\frac{d\Phi}{dt} = \frac{-c}{2L}\,(1-r_1 r_2)\,\Phi$$

The exponential decay of Φ causes a $\Delta\nu_c$ (bandwidth of cavity) $= \frac{1}{2\pi}\,\frac{c\,l\,g_{th}}{2L}$

But the limiting case is actually S.E. !!
it's a random process that's not typically tamed (easily)

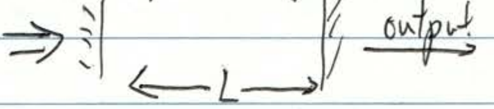
$\boxed{\text{Result}}$

$$\Delta\nu \geq \frac{\overline{N_2}}{\Delta N_t}\,\frac{h\nu_m(4\pi\Delta\nu_c)^2}{2\pi P_{out}} = \frac{\overline{N_2}}{\Delta N_t}\,\frac{8\pi h\nu_m(\Delta\nu_c)^2}{P_{out}}$$

where $\overline{N_2}$ is the avg pop^n in level 2 & $\Delta N_\ell = N_2 - N_1$

L12

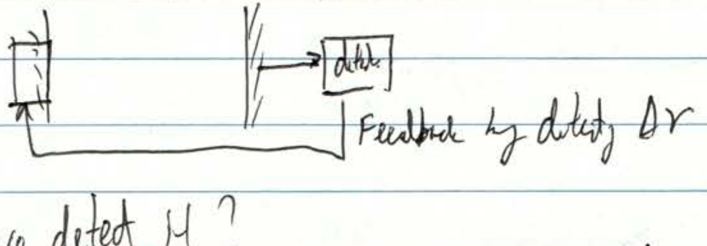
§3.4 Stabilization of Laser Frequencies



$\nu_m = \left(\frac{c}{2L}\right) m.$

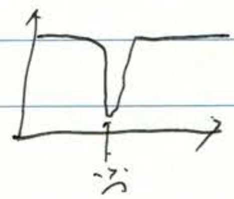
if $L = L + \delta L$ or $n = \pm\,\delta n$, etc, ν will jitter : $\nu_m + \delta\nu$
↳ (on a ms scale

We can use a PZT, or a piezoelectric transducer.



How do we detect it ?
- absorption cell
- reference F-P cavity
  ↳ R if $\nu_m \neq \nu_{cav}$

BUT $R(\nu)$ is a even fcn around $\nu = \nu_{desired}$.
can't easily control



Pound Drever Hall technique for stabilization.

$E_{out} = E_0 \exp(-i2\pi V_{out}t)$ ⇒ modulate the signal, i.e. change in time varying manner

PHASE MOD: $E_{OUT} = e^{i\phi(t)} E_{in}(t)$ AMPLITUDE MOD $E_{OUT} = A(t)\, E_{in}(t)$

↓

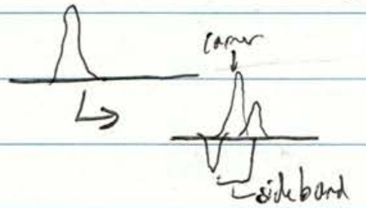
e.g. $\phi(e) = \beta \sin(2\pi f t) \Rightarrow E_{out} = E \exp\left[-i2\pi \nu_{out} t - i\beta \sin(2\pi f t)\right]$

$$e^{i\beta \sin t} = \sum_{n=-\infty} J_n(\beta)\, e^{inx}$$

or taylor here $t=0$

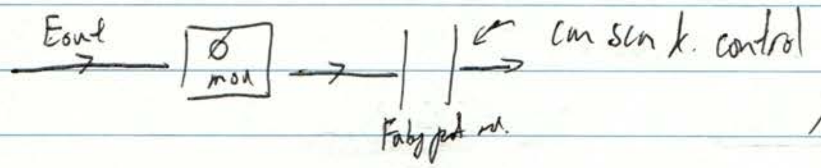
$$\hookrightarrow e^{i\beta \sin(t)} \approx 1 - i\beta \sin(2\pi f e) = 1 - \frac{\beta}{2} e^{i2\pi f t} + \frac{\beta}{2} e^{-i2\pi f e}$$

2 phase shifts w/ opposite signs!!!



$$E_{PM\,out} = E_0 \left\{ \exp(-i2\pi \nu_s t) - \frac{\beta}{2} \exp(-i2\pi(\nu_{out} - f)t) \right) + \frac{\beta}{2} \exp(-i2\pi(\nu_{out}+f)t)$$

CONTROLLED BY,



, Assuming, $f >> \delta_{FP}$

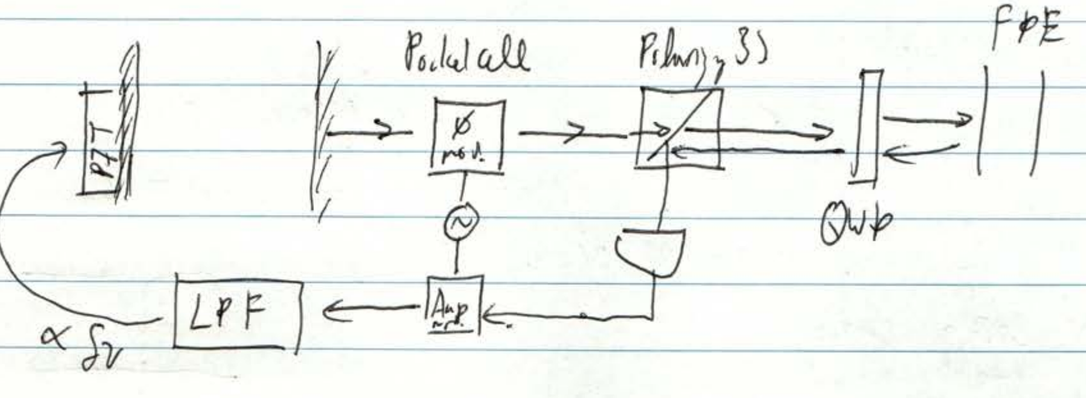
$$E_R = E_0 \left\{ r_e(\nu_{out})\, e^{-i2\pi\nu_{out}} - \frac{\beta}{2} e^{-i2\pi(\nu_{out}-f)e} + \frac{\beta}{2} e^{-i2\pi(\nu_{out}+f)e} \right\}$$

$I_R \propto |E_R|^2$.

$$\approx I_0 \left\{ |r_e(\nu_{out})|^2 + \cos^2(2\pi f t) + 2\beta.\, I_m\{r_e(\nu_{out})\} \times \sin(2\pi f e) \right\}$$

when. $I_m\{r_e(\nu_m + \delta\nu)\} \approx \frac{1}{1-r^2} \frac{d\nu}{\Delta\nu}$

Setup:



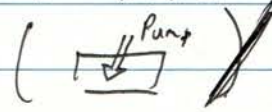
§ 5.5 Pulsed Laser Operation

Q switching!

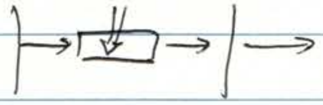
↳ Q: Quality Factor of Laser resonance.

$$Q = \frac{\text{freq}}{\text{bandwidth}} = \frac{\nu_m}{2\delta\nu_c}$$

A Q switched lasers:

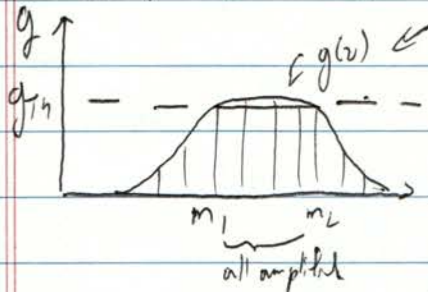


low Q ⇒ Very high pop^n inversion, w/ strong pumping, g_{th} high.



high Q it all goes out, mode locked, transient effect

MODE LOCKING.



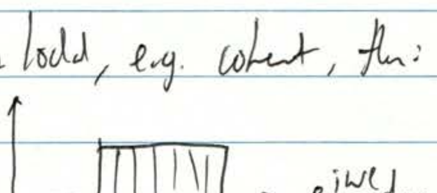
← homogeneously broadened

$$\text{out signal} = \sum_{m=m_1}^{m_2} E_0 \,\mathrm{Re}\left\{\exp\left(i2\pi\Delta\nu t m + \phi_m\right)\right\}$$

↑ if $\phi_m = \phi_0$ locked.

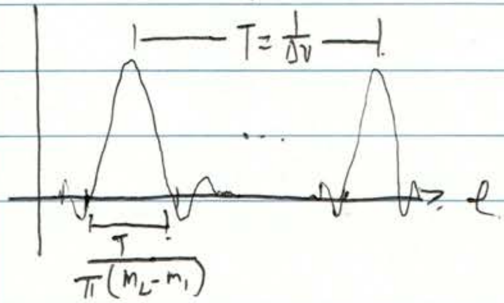
if modes locked, e.g. coherent, then:

freq:



How do we lock?
↳ num methods
(will

Time.



PHY485 Unit 4: Propagation Theory

4.1 Scalar Diffraction Theory

Using Maxwell's Eqⁿs w/ $\rho_f = 0$, $\vec{J}_f = 0$, $\vec{M} = 0$ $\vec{P} = 0$
We can treat $\vec{E} = \text{Re}\{\vec{e}\,U(\vec{r})e^{-j\omega_0 t}\}$ as $U(r,t) = \text{Re}\{U(\vec{r})e^{-j\omega_0 t}\}$
For any a monochromatic bass wave. ↳ pol vector (unit)

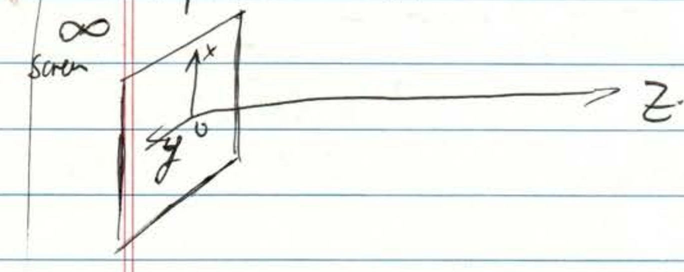
Caveats: poor assumption of scalar, hard to justify, & $\nabla \cdot \vec{E} = 0$ is ~ true

From Maxwell's eqⁿs & some math to obtain

$$(\nabla^2 + k_0^2)U(\vec{r}) = 0 \qquad \text{[Homogeneous Helmholtz Eqn]}$$

↳ where $K_0 = \underset{\text{free space}}{\omega/c} = \omega_0\, n(\omega_0)/c$

The problem is thus:



- Assuming we know $U(x,y,0)$, find. $U(x,y,z)$,
- Typically solved w/ Huygens, but that's an ad-hoc solⁿ

Solve using green's fcns

spatial $\rightarrow (\nabla^2 + k_0^2)\underbrace{g(\vec{r}-\vec{r}\,')}_{\text{green's fn sol}^n\text{ to } U(r)} = \delta^{(3)}(\vec{r}-\vec{r}\,')$

$$\text{IFT in 4D}: \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(r-r', t-t') = \delta^{(3)}(\vec{r}-\vec{r}\,')\delta^{(1)}(t-t')$$

$$G(\vec{r}-\vec{r}\,', t-t') = -\frac{1}{4\pi}\frac{\delta(t-t'-R/c)}{R}$$

$$\boxed{g(\vec{r}-\vec{r}\,') = -\frac{1}{4\pi}\frac{e^{ik_0R}}{R}}$$

Divergence Thm (Gauss's Theorem)

$$\int_V (\nabla\cdot\vec{F})\, d^3r = \oint_S \vec{F}\cdot d\vec{\sigma}$$

choose $\vec{F} = u\nabla g - g\nabla u.$

$$\nabla\cdot\vec{F} = \nabla u\nabla g + u\nabla^2 g - \nabla g\nabla u - g\nabla^2 u$$

$$= u\nabla^2 g - g\nabla^2 u$$

Then: $$\int_V (\underset{\uparrow}{u\nabla^2 g} - \underset{\uparrow}{g\nabla^2 u})\, d^3r = \oint (u\nabla g - g\nabla u)\cdot d\vec{\sigma}$$

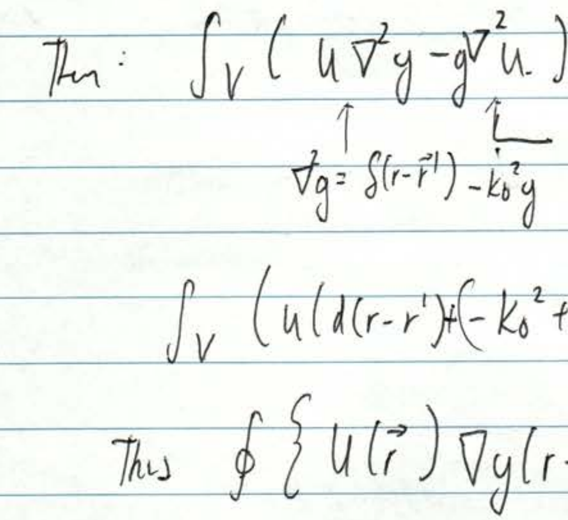
$\nabla^2 g = \delta(r-\vec{r}\,') - k_0^2 g$ $\qquad$ $\nabla^2 u = -k_0^2 u$

$$\int_V \left(u\,\delta(r-r') + (-k_0^2 + k_0^2)gu\right) d^3r = \oint (u\nabla g - g\nabla u)\cdot d\vec{\sigma}$$

Thus $$\oint \left\{ u(\vec{r})\nabla g(r-r') - g(r-r')\nabla u(\vec{r}) \right\}\cdot d\vec{\sigma} = \begin{cases} u(r') & \text{if } r'\in V \\ 0 & o/w \end{cases}$$

↖ Hyugens!!!

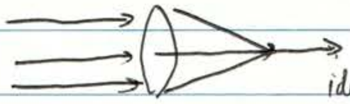
We can solve this for the Hemispherical geometry.



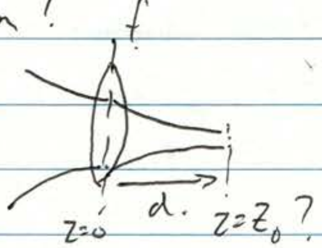
in order to use it for the propagation problem, we specify that the shell does not contribute, i.e. the plane of $z=0$ is the source.

as $R\to\infty$, solves our problem. called the shell radius

What is the spot size of a focused Gaussian beam?



ideal v.s. gaussian



Assume incident beam has its waist @ the lens:

$\hookrightarrow \; q_i = -i\frac{\pi}{\lambda}w_0^2 = -iz_0$

ABCD:

$$\begin{pmatrix}1&d\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-\frac{1}{f}&1\end{pmatrix}=\begin{pmatrix}1-\frac{d}{f}&d\\-\frac{1}{f}&1\end{pmatrix}$$

$$q_f=\left(\frac{(1-\frac{d}{f})q_i+d}{q(-1/f)+1}\right)=\frac{-iz_0(1-d/f)+d}{iz_0/f+1}=\left[\frac{1}{R(d)}+\left(\frac{iW_0^2}{z_0W(d)^2}\right)\right]^{-1}$$

$$\frac{1}{R(d)}+\frac{iW_0^2}{z_0W(d)^2}=\frac{1+iz_0/f}{d-iz_0(1-d/f)}=\frac{(1+i\frac{z_0}{f})(d+iz_0(1-d/f))}{d^2+z_0^2(1-d/f)^2}$$

$$=\frac{d-z_0^2(1-d/f)\frac{1}{f}}{d^2+z_0^2(1-d/f)^2}+\frac{iz_0}{d^2+z_0^2(1-d/f)^2}$$

$$R(d)=\frac{d^2+z_0^2(1-d/f)^2}{d-z_0^2(1-d/f)\frac{1}{f}}\qquad\frac{W(d)^2}{W_0^2}=\frac{d^2}{z_0^2}+(1-d/f)^2$$

$$\boxed{W_0^{new}=w_0^{old}\frac{f/z_0}{\sqrt{1+(f/z_0)^2}}}$$

waist occs when: $R\to\infty$

$$d-\frac{z_0^2}{f}(1-d/f)=0$$

$$d=\frac{z_0^2}{f}\left(1-\frac{d}{f}\right)$$

$$d\left(1+\frac{z_0^2}{f^2}\right)=\frac{z_0^2}{f}$$

$$d=\frac{z_0^2}{f}\Big/\left(1+\frac{z_0^2}{f^2}\right)\boxed{=\frac{f}{\frac{f^2}{z_0^2}+1}}$$

If. $\lambda=6320\,\text{Å}$ $\quad f<<z_0$

$f\approx10\,\text{cm}$ $\quad w_{old}=1\,\text{mm}$

$$W_{new}=2\times10^{-4}f$$
$$=2\times10^{-5}\,\text{m}$$

Higher order Transverse Modes

we're done: $\mathcal{E}_0(\vec{r}) = A \exp\left[\frac{ik(x^2+y^2)}{2q}\right] \exp\left[ip(z)\right]$

Instead: $\mathcal{E}_0(\vec{r}) = A\, g\left(\frac{x}{w(z)}\right) h\left(\frac{y}{w(z)}\right) \exp\left[\frac{ik(x^2+y^2)}{2q}\right] \exp\left[ip(z)\right]$

thus: g & h are sol^ns

$$\frac{\partial^2 f}{\partial u^2} - 2u\frac{\partial f}{\partial u} + 2nf = 0$$

if f, g, bounded, $\in \mathbb{Z}$

FULL H-O Sol^n:

$$\mathcal{E}(\vec{r}) = A\frac{w_0}{w(z)} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) H_m\left(\frac{\sqrt{2}y}{w(z)}\right) \exp\left[i\left\{kz - (n+m+1)\tan^{-1}\left(\frac{z}{z_0}\right)\right\}\right] \exp\left[\frac{ik(x^2+y^2)}{2R(z)}\right] \exp\left[\frac{-(x^2+y^2)}{w(z)}\right]$$

where $H_{n,m}$ are the Hermite Polynomials

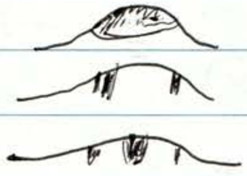
E.g. $TEM_{n,m}$

00

01

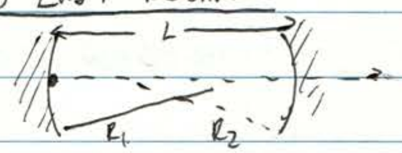
02

12

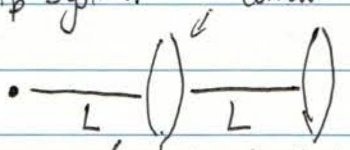


etc.

§ 4.5 Laser Resonator



Round trip System. concave mirror == convex lens w/ $f = \frac{R}{2}$



$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-2}{R_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-2}{R_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{2L}{R_2} & 2L \frac{2L^2}{R_2} \\ \frac{4L}{R_1R_2} - \frac{2}{R_1} - \frac{2}{R_2} & 1 - \frac{2L}{R_2} - \frac{4L}{R_1} + \frac{4L^2}{R_1R_2} \end{pmatrix}$$

For N round Trips

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^N = \frac{\sin N\theta}{\sin \theta} \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \frac{\sin (N-1)\theta}{\sin \theta} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

assuming $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1$ & $\cos \theta = \frac{A+D}{2}$

Thus cavity stability is: $|A+D| < 2$ for real angle: becomes hyperbolic o/w

$$\frac{1}{2}(A+D) = 1 - \frac{2L}{R_2} - \frac{2L}{R_1} + \frac{2L^2}{R_1R_2}$$

define $g_1 = 1 - \frac{L}{R_1}$, $g_2 = 1 - \frac{L}{R_2}$

$$g_1 g_2 = 1 - \frac{L}{R_1} - \frac{L}{R_2} + \frac{L^2}{R_1R_2} ;$$

$$\frac{A+D}{2} = 2g_1g_2 - 1 ;$$

$$\boxed{0 < g_1 g_2 < 1}$$ Stability Conditions

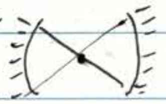
Examples

$R_1 = R_2 = \infty$ $\qquad g_1 = g_2 = 1$ (quasi-stable)

Parallel

spherical concentric

$R_1 = R_2 = \frac{L}{2}$



$g_1 = g_2 = 1 - \frac{L}{L/2} = -1 \Rightarrow g_1 g_2 = 1$ (quasi stable

Unstable!

$R_2 = \infty \qquad R_1 = \frac{-1}{R}$ $\qquad$ unstable!

$g_2 = 1 \qquad g_2 = 1 + \frac{L}{R} > 1$:

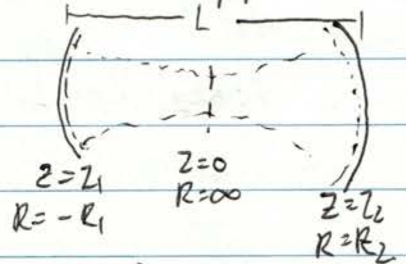
Gaussian Beams & Mods of Cavities

$$U(\vec{\rho}) = \int K_{\text{roundtrip}}(\vec{\rho}, \vec{\rho}\,')\, U(\vec{\rho}\,')\, d^2\rho'$$

$\rho = \{x, y\}$

$U(\rho')$ must be an eigen fn w/ $\lambda = 1$

Intuitive approach.



$R(z) = z + \frac{z_0^2}{z}$

↳ must be == to mirror R

$$\left.\begin{aligned} z_1 + \frac{z_0^2}{z_1} &= -R_1 \\ z_2 + \frac{z_0^2}{z_2} &= R_2 \\ z_2 - z_1 &= L \end{aligned}\right\} \text{solve for } z_1, z_2, z_0^2$$

Results

$$\boxed{z_1} = \frac{-L g_2 (1 - g_1)}{g_1 + g_2 - 2 g_1 g_2}$$

$$\boxed{z_2} = \frac{L g_1 (1 - g_2)}{g_1 + g_2 - 2 g_1 g_2}$$

$$\boxed{z_0^2} = \frac{L^2 g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2 g_1 g_2)^2}$$

$$\boxed{W_{z_2}}^2 = \frac{\lambda L}{\pi} \sqrt{\frac{g_1}{g_2 (1 - g_1 g_2)}}$$

Spotsize @ mirror 2

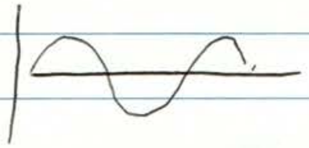
Mode Frequencies

$$\nu_{q,m,n} = \frac{c}{2L}\left(q + \frac{1}{\pi}(n+m+1)\cos^{-1}\left(\sqrt{g_1 g_2}\right)\right)$$

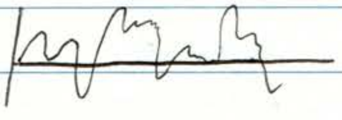
longitudinal mode index $\quad$ y $\quad$ x

PHY485: Unit 5

- While we have studied light as idealized sines & cosines, The reality is that an E/M wave is closer to random processes



→ idealized wave.



→ The actual wave is a sum of realizations (observations) of the random process

formation

★ $p(x_1, t)\, dx_1$ = the probability that the random function will take on the value $\in [x_1, x_1 + \delta x]$

$p(x_1, x_2, t_1, t_2)\, dx_1\, dx_2$ = the probability that the function ~~is~~ takes on a value $\in [x_1, x_1 + dx_1]$ @ t_1, and takes on a value $\in [x_2, x_2 + dx_2]$ @ t_2

§5.1 STATIONARY RANDOM PROCESSES

→ In these class of RP, the absolute time position is irrelevant, & the starting pt in time doesn't matter:

ie.
$$p(x_1, t) \Rightarrow p(x_1)$$
$$p(x_1, x_2, t_1, t_2) \Rightarrow p(x_1, x_2, t_2\, t_1)$$
etc.
$$p(x_1, x_2, x_3, t_1, t_2, t_3) \Rightarrow P(x_1, x_2, x_3, t_3 - t_2, t_2 - t_1)$$

Ensemble Averages

define: $x(t)$ is an RP

$F(x(t))$ is an RP as well:

$$\langle F(x(t)) \rangle = \int_{-\infty}^{\infty} F(x)\, p(x,t)\, dx$$

$$\langle F(x_1(t_1), x_2(t_2), \dots) \rangle = \int_{-\infty}^{\infty} \cdots \int F(x_1, x_2, \dots)\, p(x_1, x_2, \dots, t_1, \dots)$$

→ Average value of F for all possible states (x).

→ Hard to measure: How to realize all possible states.

→ Thus ...

TIME AVERAGES

TA → $\overline{F[{}^k x(t)]} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F({}^k x(t))\, dt.$

k^{th} realization of $x(t)$

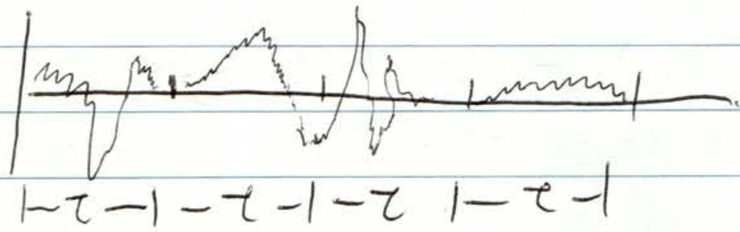
$$\overline{F[{}^k x(t_1), {}^k x(t_2)]} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F[{}^k x(t+t_1), {}^k x(t+t_2)]\, dt.$$

Same realization

How are these the same?

↳ if stationary fcns (which most things are)

↳ often can be subdivided into small intervals and each τ will represent a realization of the fcn.

↳ 

as T & τ → ∞, TA & EA become equivalent

☆ ERGODIC PROCESSES

Generalizatins to make:

↳ RPs can be complex

$$X(t) \to Z(t) = x(t) + iy(t)$$

$$p(x,t) \to p(Z,t) = p(x,y,t)$$

↳ We oft work w/ Fields insted of single processes

$$Z(t) \to Z(\vec{r},t)$$

§ 5.2. Analytic signal response / representation.

⇒ can we generalize $X(t)$ like:

$$\cos(\omega t) \to e^{i\omega t} ?$$

$$X(t) \to \tilde{x}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t) e^{-i\omega t} dt$$

but if $X(t)$ is real, then $\tilde{x}(\omega) = \tilde{x}^*(-\omega)$

↳ duplicated info, like $e^{i\omega t}$.

Then, we can multiply $\tilde{x}(\omega)$ w/ $2\textcircled{H}(\omega)$ & retain all info:

$$\boxed{\text{COMPLEX ANALYTIC SIGNAL}} : Z(t) = 2\int_0^{\infty} \tilde{x}(\omega) e^{i\omega t} d\omega$$

Properties: $X(t) = \mathrm{Re}\{Z(t)\}$

$$Z(t) = x(t) + i y(t)$$

$$y(t) = -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{x(t')}{t-t'} dt' \quad \to \text{Hilbert Transform}$$

Caucy principle part

$$= \lim_{\varepsilon \to 0^+} \left\{ \int_{-\infty}^{t-\varepsilon} \frac{x(t')}{t-t'} dt' + \int_{t+\varepsilon}^{\infty} \frac{x(t')}{t-t'} dt' \right\}$$

ADEC1

$$\langle F[x(t_1), x(t_2)] \rangle = \int_{-\infty}^{\infty} F(x)\, p(x,t)\, dx \approx \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(x_1^k(t)\, x_2^k(t))\, dt$$

5.3 Spectrum of light

~~Correlation~~ Function.

Correlation $\Gamma(\vec{r}_1, \vec{r}_2, \tau) = \langle \mathcal{E}^*(\vec{r}_2, t-\tau)\, \mathcal{E}(\vec{r}_1, t) \rangle$

if Ergodic $\Rightarrow = \overline{\mathcal{E}(\vec{r}_1, t+\tau)\, \mathcal{E}^*(\vec{r}_2, t)}$ $[t = t+\tau)]$

↑ conv of the two

General Properties:

① $\Gamma^*(\vec{r}_2, \vec{r}_1, -\tau) = \Gamma(\vec{r}_1, \vec{r}_2, \tau)$

② $\sum_{n,m=1}^{N} a_n a_m^* \Gamma(\vec{r}_n, \vec{r}_m, t_n - t_m) \geq 0$ [non negative, definite fcn]

↳ $a_1, a_2 \ldots a_N$

③ $\langle | \sum_{n=1}^{N} \mathcal{E}(\vec{r}_n, t_n)\, a_n |^2 \rangle \geq 0$

④ $\Gamma(\vec{r}_1, \vec{r}_1, 0) \geq 0$

$|\Gamma(\vec{r}_1, \vec{r}_2, t)|^2 \leq \Gamma(\vec{r}_1, \vec{r}_1, 0)$

↳ $\leq \Gamma(\vec{r}_2, \vec{r}_2, 0)$

Simple use case:

↳ $Z(t) = \xi\, e^{-i\omega t}$ where $\xi \in \mathbb{C}$, RP w/ $\langle \xi \rangle = 0$

Thus $\Gamma(\vec{r}, \vec{r}, \tau) = R(\tau) = \langle Z(t+\tau)\, Z^*(t) \rangle$

↑ Z(t) not a field.

$= \langle |\xi|^2 e^{-i\omega\tau} \rangle = \langle |\xi|^2 \rangle\, e^{-i\omega\tau}$

↑ ensemble avg of $|\xi|^2$ ↑ stationary!!

Slightly more complex

choose $Z(t) = \xi_1 e^{-i\omega_1 t} + \xi_2 e^{-i\omega_2 t} + \dots + \xi_m e^{-i\omega_m t}$

$\uparrow \quad \uparrow \quad \uparrow$ Zero mean. complex rand vars

thus $$R(\tau) = \left\langle \sum_{jk}^{n} \xi_j \xi_k^* e^{-i\omega_j (t+\tau)} e^{j\omega_k t} \right\rangle$$

$$= \sum_{j,k} \langle \xi_j \xi_k^* \rangle e^{-i(\omega_j - \omega_k)t} e^{-i\omega_j \tau}$$

$\uparrow$ dependence on t!!

↳ not stationary

UNLESS: $\langle \xi_j \xi_k^* \rangle = 0$: i.e. for a multi chromatic RP fn to be stationary, diff ω_j, ω_k, $j \neq k$ must be uncorrel.

$$R(\tau) = \sum_k \langle |\xi_k|^2 \rangle e^{-i\omega_k \tau}.$$

* * *

Some issues: if $\xi(\vec{r}, t) = Z(t)$

$$\tilde{Z}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(t) e^{i\omega t} dt. \quad \text{DNE.}$$

Why? we know that

$$R(0) = \langle |Z(t)|^2 \rangle \geq 0 \text{ \& is finite.}$$

Thus $$0 < \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |Z(t)|^2 dt < \infty$$

$\frac{1}{\infty}$ $\uparrow$ $\quad$ $\uparrow$ must be ∞

not part of L^2 fns: FT does not exist. ☹

But: if $\langle \tilde{Z}(\omega_1) \tilde{Z}^*(\omega_2) \rangle = S(\omega_1) \delta(\omega_1 - \omega_2)$

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{i\omega\tau} d\tau.$$

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{-i\omega\tau} d\omega$$

auto correlation fun ↗ $\qquad$ ↑ power spectrum

~~A~~ Weiner & Khitchin

M Dec 5

Summary of last time: $E(r,t)$ is the optical field.

→ is an RP — i.e. @ each instant it is a rand variable

→ zero mean. — stationary

Most observable quantities are derivable to the correlation/coherence fns; which is:

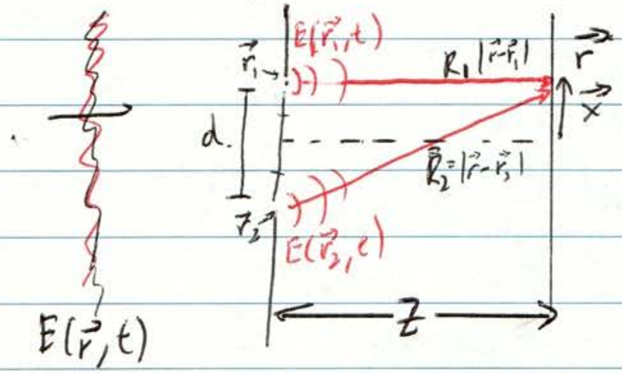
$$\Gamma(\vec{r}_1, \vec{r}_2, \tau) = \langle E(r_1, t+\tau)\, E^*(\vec{r}_2, t)\,)$$

e.g. Average Intensity $I(\vec{r}) = \Gamma(\vec{r}, \vec{r}, 0)$

Power Spectrum $S(\vec{r}, w) = \frac{1}{2\pi}\int \Gamma(\vec{r}, \vec{r}, \tau)\, e^{-iw\tau}\, d\tau$

↳ (Wiener Khitchin Thm)

S.4. Youngs Interference Experiments



$$E(\vec{r}, t) \propto \frac{1}{R_1} E\left(\vec{r}_1, t - \frac{R_1}{c}\right) + \frac{1}{R_2} E\left(\vec{r}_2, t - \frac{R_2}{c}\right)$$

auto correlation fns

Thus: $$I(\vec{r}) \propto \frac{1}{R_1^2}\left\langle E\left(\vec{r}_1, t-\frac{R_1}{c}\right) E^*\left(\vec{r}_1, t-\frac{R_1}{c}\right)\right\rangle + \frac{1}{R_2^2}\left\langle E\left(\vec{r}_2, t-\frac{R_2}{c}\right) E^*\left(\vec{r}_2 \cancel{R_2}, t-\frac{R_2}{c}\right)\right\rangle$$
$$+ \frac{1}{R_1 R_2}\left\{\left\langle E\left(\vec{r}_1, t-\frac{R_1}{c}\right) E^*\left(\vec{r}_2, t-\frac{R_2}{c}\right) + E^*\left(\vec{r}_1, t-\frac{R_1}{c}\right) E\left(\vec{r}_2, t-\frac{R_2}{c}\right)\right\rangle\right.$$

$$I(\vec{r}) \propto \frac{1}{R_1^2}\Gamma(\vec{r}_1, \vec{r}_1, 0) + \frac{1}{R_2^2}\Gamma(\vec{r}_2, \vec{r}_2, 0) + \frac{2}{R_1 R_2}\,\mathrm{Re}\left\{\Gamma\left(\vec{r}_1, \vec{r}_2, \frac{R_2 - R_1}{c}\right)\right\}$$

Alternately:

$I_1(\vec{r})$ = Intensity w/ pinhole 2 blocked @ $\vec{r}$

$I_2(\vec{r})$ = Intensity w/ pinhole 1 blocked @ $\vec{r}$

$I_1(\vec{r}) \propto \frac{1}{R_1} \Gamma(\vec{r}_1, \vec{r}_1, 0)$, etc.

$$\boxed{I(\vec{r}) = I_1(\vec{r}) + I_2(\vec{r}) + 2\sqrt{I_1(\vec{r})}\sqrt{I_2(\vec{r})}\, \mathrm{Re}\left\{ \gamma\left(\vec{r}_1, \vec{r}_2, \frac{R_2-R_1}{c}\right)\right\}}$$

complex degree of coherence

where $$\gamma(\vec{r}_1, r_2) = \frac{\Gamma(\vec{r}_1, \vec{r}_2, \tau)}{\sqrt{\Gamma(\vec{r}_1, \vec{r}_1, 0)\, \Gamma(\vec{r}_2, \vec{r}_2, 0)}}$$

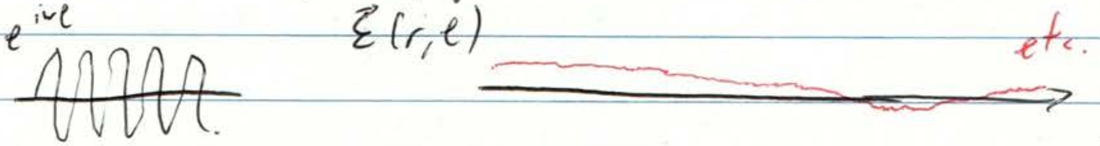
$$|\gamma(\vec{r}_1, \vec{r}_2, \tau)| \leq 1 \qquad |\gamma(\vec{r}, \vec{r}, 0)| = 1$$

Investigating an approx of reality: QUASI-MONOCHROMATIC FIELD

$$E(\vec{r}, t) = \underbrace{\mathcal{E}(\vec{r}, t)}_{} e^{-i\omega_0 t}$$

a slowly varying random variable:

e.g.

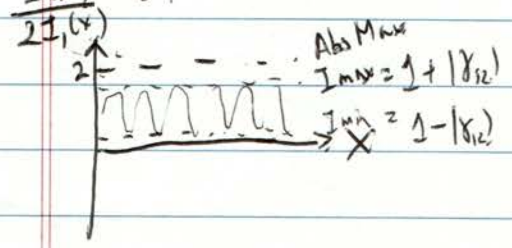


Then: $$\Gamma(\vec{r}_1, \vec{r}_2, t) = \langle \mathcal{E}(\vec{r}_1, t+\tau)\, \mathcal{E}^*(\vec{r}_2, t)\rangle e^{-i\omega t}$$

We can then assert: $R_2 - R_1 \approx \frac{xd}{z}$ $\qquad I_1(\vec{r}) \approx I_2(\vec{r})$

Then $$I(x) = 2I_1(x)\left\{1 + |\gamma_{12}| \cos\left(\frac{\omega_0 x d}{cz} + \alpha_{12}\right)\right\}$$

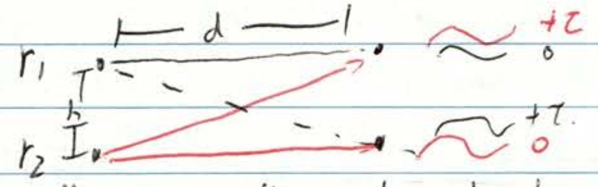
↑ phase



$$\Rightarrow V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = |\gamma_{12}|$$

Application: Michelson, measuring stars.

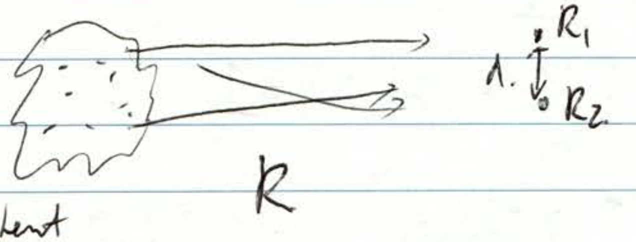
↳ since $\Gamma = \langle E\, E^{*} \rangle$ etc,



$+\tau$ will be negligible if $\frac{d}{c} >>> \tau$

→ after some math, it can be shown that propagation tends to generate coherence btwn fields

THUS.



: γ_{12} of the resulting interference is found to be $\approx \widetilde{I}_0\left(\frac{kd}{R}\right)$

Thus, if you measure γ_{12} for any R_1, R_2, you can measure all of $\widetilde{I}_0$, take the IFT, & we got $I_0(x,y)$! & can see the star